We need some terminology to get started. **Interest** is the payment received from lending money. It is an absolute sum, such as $50.00. **Return** is what you receive from any type of investment, including a loan. Return on an investment may thus include interest, but it would also include dividends on stock, or rental on a house. It will also include any change in the price of the investment over the holding period, called **capital gain** (or **capital loss**).

When interest is calculated as a proportion of the sum lent, it is called the **rate of interest** per unit of time. The **rate of return** [or yield] on an investment is the return as a proportion of the sum invested per unit of time. The definition of a rate requires the specification of both the unit of time to which it applies, and the frequency of payment of the return(s).

The **gross, or total, return** on an investment is usually taken to mean the sum of the gains or losses over the entire period that the investment is held. The **holding period rate of return** is the total return as an average per specified unit of time and specified frequency of payment for the period that an investment was held. A standard time unit is a year, or **annual rate**, with returns paid annually. A $100 deposit made at 12.01 am on January 1st that returns $5 interest at 11.59 pm December 31st pays an annual rate of return with annual frequency of 5%. The interest rate is then 5%.

This may seem simple, but not all years have the same number of days, and although all years have the same number of months, not all months have the same number of days. When time is money, calculation of yields has to take all these anomalies into account. See Annex.

You are offered a contract that provides a guaranteed payment of $1.00 in one year. How much should you pay for it today? Or, in other words, what is the “fair value”? If you can’t answer the question, try to formulate it in a way that you can answer. For example, you could turn the question around and ask: How much it would cost to produce $1.00 in one year? That is, how could you replicate the contract that you are trying to evaluate.

One way to replicate the payment of $1.00 in one year is to buy a bond that pays $1.00 in one year. The answer to the question posed above then becomes: the fair value of a bond paying one dollar in one year. This is often called a "dollar discount bond" because it costs less than it pays out, that is, it sells at a discount.
to its full value at maturity of $1.00. In a diagram this would look like:

\[
\begin{align*}
\text{\$x} & \quad \text{\$1.00} \\
\text{Year 1} & \quad \text{Year 2}
\end{align*}
\]

The horizontal line represents the passage of time from the present to the future. Arrows going below the "time" line are negative money flows representing out payments, arrows going up from the line are positive flows representing receipts.

If the one-year interest rate is 0.1764706 cents per dollar (or 17.6471%) then a discount bond purchased at a price of $0.85 today will return $1 in a year, since your initial investment will earn 0.85 \times 0.1764706 = $0.15 in interest. The repayment of the $0.85 plus the $0.15 in interest gives you $1.00. Thus, the fair price for the contract is $0.85.

To recap: How did we find the price of $1 in one year? Formally, it is simply the reciprocal of \(1 + \text{the rate interest payable on money loaned for one year}\) or \(1/(1.176471) = $0.85\). This ratio is often called the discount factor, but it is easier to think of it as the present or spot price of the one-year forward delivery of a dollar.

What we did was to solve a problem of the following form. What amount, \(x\), invested for 1 year at 17.6471\%, gives $1.00? Solving \(1 = x + 0.176471x\), for the value of \(x\) gives

\[1 = 1.176471x \quad \text{or} \quad x = 1/(1.176471) = 85 \text{ cents.}\]

That was pretty easy. How about $1.00 in two years? Again, we look for a replicating portfolio. We already know how much to pay for $1 in one year, so we could take that contract and then reinvest the dollar for
another year. If the rate of interest for one year is expected to remain unchanged, then we would have $1.176471 at the end of two years. But, that is 17.6 cents too much. We want $1.00 in two years. What we need is a contract in one year's time that pays $1 after one year starting one year from now (which is two years from now). If one-year interest rates are expected to remain unchanged, then we need to invest 85 cents at the end of one year to receive $1 at the end of year two.

But, this means that we now have to find the amount that we have to invest today to yield $0.85 at the end of the first year. This is just the fair price of 85 cents in one year. If the price of $1 in one year is 85% of 1.00, then the fair price of .85 should be 85% of the desired value or .85 * .85 = .7225 or 72.25 cents. We could then make an investment of 72.25 cents today to receive 0.7225 * 1.176471 = $0.85 in one year which can then be reinvested for the second year at the annual rate of 17.6471% to produce the desired $0.85 * 1.176471 = $1.00. Thus the fair price of $1.00 in two years is 72.25 cents.

(Note that the two $0.85 flow arrows cancel out since the arrow going up represents the receipt of the money invested at the beginning of the year, and the arrow down is the purchase of the discount bond at the beginning of year 2).

This is just a series of one-year investments set up to produce $1 at the end of the second year. Let us calculate the returns over each one-year period. Return is given by the difference between the present cost, \( PV \), and the future value, \( FV \), received from the investment. The rate of return, or yield, is the return as a proportion of the original investment:

\[
\text{RETURN, } R = (FV-PV)
\]

\[
\text{RATE OF RETURN, } r = (FV-PV)/PV
\]

which can be rewritten as

\[
r = (FV/PV) - (PV/PV) \text{ or } (FV/PV) - 1
\]

For year one we have: \((0.85 - 0.7225)/0.7225 = 0.1275/0.7225 = 0.176471\)

For year two we have: \((1.0 - 0.85)/0.85 = 0.15/0.85 = 0.176471\)

For the two years as a whole we have: \((1.0 - 0.7225) / 0.7225 = 0.384083\).
Note that this is not equal to 2 * 0.176471. Nor would it be correct to calculate the "average" return over the two-year period as 0.384083 / 2 = 0.19204 since it is clear that the rate for each of the two periods is 0.176471; the average rate cannot be anything different.

The arithmetic average return is given by 0.5 * 0.176471 + 0.5 * 0.176471 = 0.176471. However, even this is not a generally correct method for it holds only in the case of constant interest rates for every period through time. The correct method would use the "geometric mean" to calculate the average return. The geometric mean is the average rate applied to the PV (present value) per period to produce the FV (future value) resulting from the actual rates of interest applied in each period.

Thus the future value at the end of period one is:

\[ FV_1 = PV \times (1 + r_1), \]

and after two periods is:

\[ FV_2 = FV_1 \times (1 + r_2) \]

Or, by substitution:

\[ FV_2 = PV \times (1 + r_1)(1 + r_2), \]

since the rate, \( r \) is the same in both periods this reduces to:

\[ FV_2 = PV \times [(1 + 0.176471)(1 + 0.176471)] = PV \times (1.176471)^2. \]

This produces the geometric mean rate by solving for \( gr \): 

\[ (FV_2/PV) = [PV \times (1 + gr) \times (1 + gr)]/PV, \]

where is \( gr \) is the uniform average periodic rate.

\[ (1.0 / 0.7225) = 1.0 \times [ (1 + gr) \times (1 + gr)] \times 1.0 \times (1 + gr)^2 \]

\[ [(1.0/0.7225) -1] = 1.0 \times (1 + gr) \]

--Hint: remember that taking the square root of both sides is equivalent to raising the expression to 0.5 (and the cube root to 1/3, etc.) Solving gives

\[ (1.38408)^5 = 1.176471 \] or \( gr = 1.176471 - 1 = 17.6471\% \).

Note that the equality we have discovered between the arithmetic and geometric average is a special case which holds only when the interest rate remains constant period by period. If the rate of interest in year 1 had been 10%, and the interest rate in year 2 had been 20%, then an investment to yield $1 in two years would have required an investment of \( 1/1.2 = 0.8333 \) at the end of year 1. To get .833 at the end of year 1 when the interest rate in year one is 10% would have required 0.9090 * 0.8333 = .75757.

In this case we start with a PV = 0.7575 multiplied by 1.10 to yield 0.833, which is multiplied by 1.20 to give $1 at the end of the second year. This is:

\[ 0.7575(1.10)(1.20) = 1.0 = FV. \]

The geometric mean is then:

\[ 0.7575 (1.gr)^2 = 1 \]
or Return:

\[ \frac{FV}{PV} = \frac{1.0}{0.7575} = (1. gr)^2 \]

Or

\[ (1.32)^5 = 1. gr = 1.1489 \]

as the geometric average rate.

\[ r = \frac{FV}{PV} - 1 = 0.1489. \]

The average return given by the arithmetic mean would be 0.15.

**Finding the Fair Value of Fixed Interest Coupon Bonds**

Securities which pay interest, usually in the form of semi-annual or annual coupons, at a rate which is calculated by the coupon as a proportion of their principal or face value are called "interest bearing" or coupon securities. We will return to these problems caused by the interest "basis" below.

There are a number of financial instruments which are priced at a "discount" from their face value which is paid at maturity. Most are "money market" instruments with a maturity of one year or less. The most common is a Treasury Bill, which may be purchased for maturities of 3, 6 and 12 months. They are sold at a discount from their par or face value at maturity. The interest earned is represented by the difference between the purchase price and the par value, or the amount of the discount. The interest rate is thus the discount as a proportion of purchase price. The calculation of their present values is a bit more complicated than the examples we have just given.

There is one very important use for the answer to the question of the value of a $1 payment in a year, or two years, that we have now answered. It allows us to answer another question: What is the value of a stream of payments of $1 over a period of years? The answer is straightforward. It is the sum of the fair values of each of the $1 payments in the stream. A steady stream of constant payment amounts is called an "annuity". You will buy one to have an income when you retire. Here we need them to carry out valuation of fixed income securities. Such securities, usually called bonds, are interest bearing securities which pay constant amounts per period called a coupon. At the end of the period of the loan, or the maturity of the bond, the lender or investor in the bond also receives the return of principal, that is, the bond repays to the holder the par or face value. The stream of individual cash flows for the bond can thus be divided into the stream of individual coupon payments, which represent a fixed term annuity, and the single final payment
representing the return of principal. To value the income stream of a bond we thus have to value the annuity portion and then add the value of the final principal portion. Let’s start with the annuity. Consider the diagram of flows of a two-year annuity:

From the exercise above, we know that if the one-year interest rate is 0.176471, then the present value of the first dollar you receive is 85 cents. The present value of the second is 0.7225 and the fair value today of the whole two-year stream is just $0.85 + $0.7225 or $1.5725.

For annuities covering longer time periods-- up to infinity! -- we need a formula to make the calculations a little more quickly.

Define $oP_1$ as the price of $1$ in one year (0.85) and $oP_2$ as the price of $1$ in two years (0.7225) and $1P_2$ as the price of $1$ in one year, one year from now (0.85) and $A_2$ as the value of a two-year annuity. In our simple example of a two-year annuity we then have

$$oP_1 = 1P_2$$

and

$$oP_2 = oP_1 \cdot oP_1 = (oP_1)^2.$$

Now, remember the derivation of the income multiplier used in Keynesian macroeconomics as a converging infinite series? Let's do the same thing and multiply the valuation of the two-year annuity by $oP_1$ to produce

$$A_2 \cdot oP_1 = (oP_1)^2 + (oP_1)^3.$$

Now, subtract this from the original value of $A_2$ we started with:

$$A_2 - (A_2 \cdot oP_1) = oP_1 + (oP_1)^2 - (oP_1)^2 - (oP_1)^3.$$

Simplify the expression to give

$$A_2 (1 - oP_1) = oP_1 + (oP_2)^2 - (oP_2)^2 - (oP_1)^3.$$

$$A_2 = \left[ oP_1 - (oP_1)^3 \right] / (1 - oP_1)$$

$$A_2 = \left\{ oP_1 [1 - (oP_1)^2] \right\} / (1 - oP_1).$$

Let's stop a minute and see what this gives in terms of our example:
so we are still on the right track. We can simplify a bit more by multiplying by \((1 \div \alpha P_1)\) to give:

\[
A_2 = [1 -\alpha P_2]/[(1/\alpha P_1)-1] = (1-.7225)/.176471 = .2775/.176471 = $1.5725
\]

If the annuity lasted for \(n\) periods we would then have

\[
A_n = [1 -\alpha P_n]/[(1/\alpha P_1)-1].
\]

Remember that we calculated the \(P_s\)'s in terms of the fair value today of a dollar at the end of the relevant period. This means taking the reciprocal of the interest rates for the appropriate future dates:

\[
\alpha P_n = \{1/(1+i_n)^n\}
\]

\[
\alpha P_1 = \{1/(1+i_1)^1\}.
\]

Making the appropriate substitutions:

\[
A_n = [1 -\{1/(1+i_n)^n\}]/[(1/{1/(1+i_1)})-1]
\]

\[
= [1 -\{1/(1+i_n)^n\}]/\alpha i_1 \text{ (which is also written } 1/i_1\{1-[1/(1+i_n)^n]\} \text{ )}
\]

\[
= [1-(1.176471)^{-2}]/.176471 = (1 - .7225)/.176471 = 1.5725
\]

An annuity of $1 a year for 5 years would then have a fair value of:

\[
[1-(1.176471)^{-5}]/.176471 = (1 - .443705)/.176471 = $3.15
\]

An annuity of $1 a year for 10 years would then have a fair value of

\[
[1-(1.176471)^{-10}]/.176471 = (1 - .1968737)/.176471 = $4.55
\]

Not convinced? Work out the schedule of payments as follows. If you are the seller of the annuity you receive the $4.55 in exchange for the commitment to pay the buyer the sum of $1 at the end of every year for ten years. That’s a total of $10, which is a lot more than the $4.55 you received. How are you going to do it? Well, you have the payment now, and you don’t have to pay out anything for a year, so if you invest the $4.55 received today you have the following cash flows in year 1:

\[
4.551038 \text{ invested at 17.6471% produces } 5.354164.
\]

But, you have to pay the $1 annuity, so you have left $4.354164 which you invest again.

At the end of year 2 the 4.354164 invested at 17.6471% produces $5.122548.

Again you have to deduct the annuity payment to $1 and you have $ 4.122548 to invest for Year 3. Following the rest of the annual cash flows you have

\[
4.122548 \text{ invested at 17.6471% produces } 4.850058 \text{ - $1 which leaves } 3.850058 \text{ to invest in year 4}
\]

\[
3.850058 \text{ " } 4.529482 - $1 \text{ " } 3.529482 \text{ “ } 5
\]

\[
3.529482 \text{ " } 4.152333 - $1 \text{ " } 3.152333 \text{ “ } 6
\]

\[
3.152333 \text{ " } 3.708629 - $1 \text{ " } 2.708629 \text{ “ } 7
\]

\[
2.708629 \text{ " } 3.186623 - $1 \text{ " } 2.186623 \text{ “ } 8
\]

\[
2.186623 \text{ " } 2.572499 - $1 \text{ " } 1.572499 \text{ “ } 9
\]
This is the last year! You invest 85 cents and receive $1.00 which is just what you need to meet the final annuity payment with nothing left over.

Notice the importance of being able to count on the interest rate remaining stable over the life of the annuity, or being set for all the reinvestments from the beginning of the annuity contract. If the rate of interest had gone down over the life of the contract, you would not have been able to earn enough to meet the payments. In that case it might have been better to have invested for the entire 10 year period to produce a stream of $1 payments.

Some short cuts for annuity calculations.

A perpetually annuity has a return which is given by $r = C/PV$, where $C$ is the constant payment per period. This gives the present value of a perpetually annuity as $C/r$. A fixed term or finite annuity can be created by constructing a replicating portfolio using two perpetually annuities. Buy one perpetually annuity starting today and sell another starting at some future date, $t$. The value today of a perpetually annuity starting at some future date, $t$, will be given by $\frac{1}{(1+r)^t} \cdot \frac{C}{r}$. We can thus define the fair value of an annuity that starts today and runs until time $t$ as $\frac{C}{r} - \frac{1}{(1+r)^t} \cdot \frac{C}{r}$. This simplifies to

$$A_n = C\left(\frac{1}{r} - \frac{1}{1+r}^t\right).$$

From time $t$ onwards the in payments from the first annuity just cancel the out payments from the second, so the annuity terminates from that date. The value of the differences of the two terms inside the curly {}.

The formula for the fair value of a finite annuity given in Box 3 is the first component of the valuation of a coupon bond. Thus, if we want to find the fair value (PV) of a fixed interest payment bond contract which pays uniform coupon interest ($C$) over a fixed period and then returns the face value (or par value or principal), $M$, at maturity we first note that the coupon payments are just an annuity of $C\times 1$ paid over the finite life of the bond, and the repayment of par value at maturity is just the fair value of $M$ times $1$ paid in $n$ years, where $n$ is the number of years to maturity.

Consider a bond with a two-year maturity with an initial issue or par value of 5.66 and paying coupons of $1$ when the interest rate is 0.176471. The fair value of the bond, or its present value, is thus given by the fair value of the fixed two year annuity paying $1$ per year, and the return of principal at the end of
year 2 or $5.66. Calculate

\[ PV = A_2 + M \cdot P_2 \]

\[ A_2 = .85 + .7225 = 1.5725 \]

\[ M \cdot P_2 = 5.66 \cdot .7225 = 4.089 \]

\[ PV = 1.5725 + 4.089 = 5.66 \]

Bond prices are usually given as a percentage of their par value. The price of this bond would be quoted as 100, since its fair or present value = Par or face value. It’s current yield would be 17.65%. This is also it’s **coupon yield** and is given by the dollar value of the coupon divided by the par value or \( 1/5.66 = 17.65\% \) (don’t worry about the rounding error - it comes from rounding off the bond price to two digits).

We could rewrite this same relation as

\[ PV = \frac{1}{1.176471} + \frac{1}{(1.176471)^2} + \frac{5.66}{(1.176471)^2} \]

\[ = .85 + .7225 + (.7225 \cdot 5.66). \]

Putting the fair value of the bond this way shows that its value is a combination of three different payment flows: $1 in one year, $1 in two years, and 5.66 times $1 in two years. In a diagram this would be:

Thus, the two-year maturity fixed-interest coupon bond is equivalent to one, one-year dollar discount bond, and one, two-year dollar discount bond and 5.66 two-year dollar discount bonds. All of these individual
discount "bonds" share the characteristic that they are not interest bearing, i.e., they do not have coupon payments. For this reason they are often called "zero coupon" bonds and their interest rates are called zero rates or spot rates. As seen above, such discount bonds have the characteristic that their fair prices, PV, are just equal to their "discount factors", \( P_n = 1/(1+\text{interest rate})^n \). You can see that they are much easier to value than a coupon bond because they do not require an annuity calculation for the interest coupon flows. We could thus calculate the fair value of a coupon bond without doing the annuity calculation by breaking the bond into its constituent parts of zero coupon bonds.

There is another reason for breaking the calculation of the bond’s fair value into separate zero coupons. The annuity method can only be applied when our special assumptions hold, i.e., the one-period interest rate is expected to remain unchanged in future periods, since the formula only uses one rate, \( r \). This is an important problem to understand. Recall our calculation of the fair value of $1 in two years' time. The rate for the first year is fixed, but the second year rate is not, so

\[ \text{PV}_2 = 1/[(1+\text{i}_1)(1+\text{i}_2)]. \]

Say that we observe that in the market $1 in two years is selling at a price of \( \text{PV}_2 = $0.70 \). Then we have:

\[ \text{PV}_2 = 0.70 = 1/[(1.176471)(1+\text{i}_2)] \]

or

\[ 0.70 \times (1+\text{i}_2) = 1/(1.176471) = 0.85 \]

Solving for \( 1+\text{i}_2 \) gives:

\[ 0.85/0.70 = 1.214285 \] or \( \text{i}_2 = 0.214285 \).

What does this mean? It says that dealers in the market are expecting the rate for one-year bonds to be sold in one year’s time will have a rate of interest of 21.43%. Thus, what we have calculated is the market expectation of the “One-year rate one year forward”, or the one-year forward rate.

But, this means that the rate is no longer uniform. What will the two-year zero or spot rate be?

\[ \text{PV}_2 = 0.70 \times 1.176471 \text{ gives } 0.82227 \text{ in year 1 and } 0.82227 \times 1.214285 = $1 \]

\[ 1.176471 \times 1.214285 = 1.428571 \text{ and } 1/1.428571 = 0.70 \]

\[ (1/0.7)^2 = 1.195229, \]

so the geometric mean return is 0.195229. This is the two-year spot rate.

Our calculation of the fair price of the bond will now also change. The annuity value will now be

\[ 0.85 + 0.70 \text{ or } $1.55. \]

The return of face value at maturity will now be \( 5.66 \times .70 = 3.962 \) for a total value of $5.512.

Since interest rates are no longer constant over the two periods, we will also have to revise our calculation of the price given by the present value discounting:

\[ \text{PV} = 1/(1.176471)+1/(1.176471)(1.214285) + 5.66/(1.176471)(1.214285) \]

\[ = .85 + .70 + (.70 \times 5.66) = $5.512. \] This is the PV of the $5.66 bond.
However, doing the calculation this way does create a problem. How do we represent the rate of interest earned from owning the bond, given that we are now using two different interest rates to discount the two coupon payments? The correct answer to this question is found by recalling that the bond is a bundle of three separate components: a one-year bond paying a dollar, a two-year bond paying a dollar and a 5.66 two-year bonds paying one dollar. It is possible to represent the rate of interest on each of these flows. The correct specification of the return on the bond would require citing all of them. However, financial markets continue to request, and furnish, a single interest rate to represent the return on bonds with fixed coupons. This single rate is called the **yield to maturity** of the bond.

The yield to maturity is calculated by finding the rate of interest which makes the present value of the bond's various periodic cash flows over its life equal to its current cost (purchase price, which may but need not, be equal to its par or face value). Thus, to find the yield to maturity we have to find a rate of discount, \( j \), that satisfies:

\[
5.512 - \left\{ \frac{1}{1+j} + \frac{1}{(1+j)^2} + \frac{5.66}{(1+j)^2} \right\} = 0
\]

where \( j \) represents a uniform, average yield or the **yield to maturity** of the bond. The value of \( j \) should again be familiar to you from macro; it is just the internal rate of return (irr) used to find the marginal efficiency of capital. Now, we know that this rate should lie between the one-year rate and the two-year rate, so we could proceed by trial and error, or we could ask a computer to do the calculation. Mine gives the irr as \( j=19.3674\% \) which is about what we would expect. Let's try doing the PV calculation with 19.3674\% as the discount rate:

\[
\begin{align*}
\frac{1}{1.193674} &= 0.83775; \\
\frac{1}{(1.193674)^2} &= 0.70183; \\
\frac{5.66}{(1.193674)^2} &= 3.9724
\end{align*}
\]

for a total of:

\[
0.83775 + 0.70183 + 3.9724 = 5.512
\]

which is the fair value price using the correct method, so that must be close to the right answer for the irr.

If you called your financial consultant and asked about buying this bond, it would probably be recommended because of its excellent yield to maturity of over 19\%, much higher than the currently prevailing one-year interest rate of 17\%. However, what will probably not be mentioned to you is that this rate is only an average return; as any average it need not correspond to any actual return for any of the three separate payment flows that constitute the actual return on the bond! Neither will you be told that while the bond has a rate over its life that is higher than the current one-year rate, this rate is in fact lower than the implicit two-year rate. Further, you will probably not be told that the bond offers no guarantee whatsoever that it will produce the cash flows required to attain the advertised 19.37\% rate of return, even if you hold it until maturity.

To see this turn the calculation around and work out the stream of future payments which the bond is guaranteed to produce:
Year 1 = $1 coupon payment,  
Year 2 = $1 coupon payment  
Year 2 = $5.66 repayment of principal  
Total = $7.66.

Do the calculation for the bond's return:

\[
(FV/PV)^{1/2} - 1 \text{ which gives } (7.66/5.512)^{1/2} - 1 = (1.3897)^{1/2} - 1 = 1.17885 - 1 = 17.885\%.
\]

This is not what you were quoted by the financial consultant who is recommending that you buy the bond. What went wrong? The difference is in how you evaluate the first $1 coupon payment. This calculation has only included simple interest. After the first $1 coupon was received, it was not immediately reinvested. Either it was kept as idle cash, or was idly spent. On the other hand, the calculations of yield to maturity assume that you reinvest the coupon interest payments as they are received. They thus continue to earn interest until the maturity date of the bond, and the interest earned is compounded over the life of the bond. Your total return from buying the bond is thus considered to include the interest which you earn by reinvesting the interest received over the life of the bond. Your total or gross earnings on the bond thus are comprised of: 1) interest (simple interest), 2) interest on interest (compound interest), and 3) the principal repayment.

The important question that this definition of total return raises is, what rate do you receive on the reinvested coupons? Who knows the future? The financial consultant certainly does not, so when the calculation of the yield to maturity is made, it is simply assumed that the current yield on the bond will continue to be earned on all funds reinvested until maturity. This means that the yield to maturity calculation assumes that you can reinvest the accrued coupons at the yield to maturity. This adds another 19.3674 cents (the interest on the first $1 coupon invested at 19.3674% for the second year) to the total future value you will receive at maturity to give 7.8536. Now do the calculation (7.8536/5.512)^{1/2} - 1 and you get (1.424832)^{1/2} - 1 = 1.19336 - 1 = 19.336% which is your quoted yield to maturity.

But, you only get that yield if you manage to reinvest your coupon payments over the life of the bond at the original yield to maturity rate and earn 19.37 cents from reinvesting the first coupon. You should be so lucky. The high probability that you will not be able to reinvest at this rate is called "reinvestment rate risk", and is simply the risk that you will not be able to invest your future interest coupons at the yield to maturity rate. If you can't, the actually received yield on the bond will be lower than the quoted yield to maturity if rates fall — too bad, or higher than the quoted yield if rates rise — you got lucky.

Is there a better way to form an expectation of the future rate at which you will be able to reinvest your coupon interest? Of course. We already know what the yield for one-year investments and the yield for
two-year investments. By combining the two we can calculate what the market is implicitly expecting the one-year rate to be in one year's time.

Let's start by drawing the diagrams for the coupon flows.

![Diagram of coupon flows](image)

Zero coupon 1 year discount bond
Zero coupon 2 year discount bond

What if we sold a one-year bond and bought a two-year bond? That would look like this:

![Diagram of coupon flows](image)

Sell Zero coupon 1 year discount bond
Buy Zero coupon 2 yr discount bond

Combining the two produces:
Today you have an inflow of 85 cents and an outflow of 70 cents for a net inflow of 15 cents. In one year you have an outflow of $1 which produces an inflow of $1 in two years' time. This is the equivalent of investing for one year, starting in one year's time. This costs you $1 in one year less 15 cents today. But, we can only make comparisons at the same point in time. Those 15 cents will be worth $0.15 \times 1.176471 = $0.17647 in one year so you really invest $1.00 - .17647 = $0.823529 in one year to get $1 the year after. This is a return of $1.823529 - 1 = 0.2142858. The complete diagram would look like:

One-year investment starting in one year.

This is what is called a "forward" transaction. It will produce a "forward interest rate", that is, the rate of interest that will apply in one year, if the contract is made today. Above it was called $i_2$, or the rate applied during the second year to produce the $1 return at the end of year 2. Recall that the one-year rate is 17.65% and the price of $1 in two years of $0.70 gives an annual rate of $(1/.7)^5 - 1 = 19.52\%$, so the 21.43% forward rate is higher. The market expects interest rates to be higher next year. Indeed, anyone entering a two-year contract would have as an alternative investing for one year and then reinvesting for a second year at the expected one-year forward rate. A $1 investment would give 1.176471 which could be reinvested at the expected forward rate for another year to give 1.176471 * 1.2142858 = 1.428572 and a two-year gain of 0.428572. As an annual return this is 1.19522 - 1 = 19.52\%. But, this is just the annual return on the two-year zero coupon investment!

Thus, we can also define the two-year zero rate as being the result of the combination of the current one-year rate and the expected one-year rate, one year ahead (i.e. to start in one year's time) or $1.1765 \times 1.2143 = \text{one-year zero rate times the expected one-year zero rate in one year's time} = \text{two-year zero rate. We say}
that the expected forward rate is "embedded" in the two-year zero rate. This is one of the reasons for working out the zero rates. But, if the one-year rate in one year is embedded in the two-year zero, the one-year rate in two years' time must be embedded in the three-year zero rate (as is the two-year rate in one year’s time), and so forth. If we have a full range of zero rates for all maturities, we can then calculate a full maturity range of forward rates. This is the basis for the "expectations" explanation of the yield differential between bonds of different maturity: The two-year rate is determined by today's rate and the expectation of rates next year, the three-year rate is determined by the two-year rate and the expectation of the one-year rate in two years (or the current rate and the expectation of the two-year rate in one year) and so forth. The entire yield curve is thus determined by the expectations of future one-year (or two-year, or three-year, etc.) rates.

We have worked out these rates on the basis of dollar discount, or zero coupon, bonds. But, bond markets usually trade only fixed coupon bonds. What if we can't find a market price for a zero coupon bond? How would we calculate the forward rates then? The answer now should be easy. We have already noted that a two-year fixed interest coupon bond can be decomposed into two zero coupon bonds. One of them is the one-year zero. We can always "replicate" a one-year zero coupon discount bond with a one-year Treasury bill. Thus we can always discover the one-year zero rate. We can use this as the basis for the calculation of the two-year zero as follows. Say our two-year bond is the one used above (and we keep the expectation of higher rates in year 2):

Before we proceed, remember that it is highly unlikely that you would ever see a bond quote like this. As we noted above, the practice is to quote the current price as a percentage of the par or face value of the bond. In this case the original face value was $5.66; this would give (5.512/5.66) or 97.39. (Even this would be abnormal since most bonds are issued with par or face values which are multiples of $1,000.)

We already know that this payment flow has a yield to maturity of 20.27%. We know that the one-year rate is 17.6471% and the price of $1 in one year is $0.85. Our problem is to "engineer" this bond into looking like a two-year zero. If we could strip off the two $1 arrows, then it would look like a zero coupon. In fact, we only have to strip one of them, since the second one occurs at the end of year two so it can be included with the
repayment to give a final payment of $6.66.

So, all we have to do is get rid of the first dollar coupon payment. The simple way is to sell it. What is the fair price of a dollar in one year? We already know that it is 85 cents. So sell the first coupon today for 85 cents, which is received today. This can be set against the cost of the bond $5.512 - 0.85 = $4.662. This is the sum that you pay to receive $6.66 in two years' time. The rate of return on this "engineered" two-year zero coupon bond is then \((6.66/4.662)^{\frac{1}{2}} - 1 = 19.52\%\) which is just the two-year zero average rate we found above.

We can proceed in this way to produce a full spectrum of zero rates from the rates on coupon bonds. This is called "bootstrapping" because we are pulling ourselves up the yield curve on the basis of our own bootstraps represented by the one-year discount bond rate.

**Reinvestment Rate Risk Again**

Let us take a par bond, selling at 100 (i.e. its market price is equal to its face value), paying a coupon of $5 on an annual basis for a period of five years. It's yield to maturity is then 5%, which is equal to the **coupon yield** \((5/100)\). Each year your money returns 5%, but each year you have $5.00 of your money that has to be reinvested. Only if you can do this at a 5% rate for each of the last four years will you in fact earn 5%. If you are not yet convinced, calculate the future value in five years of the $100 present value you invest today. The first coupon payment of $5 will be reinvested for 4 years so it will have a value of \(5(1.05)^4\) at maturity. The second coupon will have a value of \(5(1.05)^3\) and so forth. We can thus write the FV as:

\[
FV = 5(1.05)^4 + 5(1.05)^3 + 5(1.05)^2 + 5(1.05)^1 + 105 = 127.6282
\]

The return is \((127.6282/100)^{0.20} - 1 = .05\).

When we calculate the fair price of a coupon bond, i.e. its present value, all we do is to reverse this procedure and find

\[
PV = 5(1.05)^{-1} + 5(1.05)^{-2} + 5(1.05)^{-3} + 5(1.05)^{-4} + [100+5](1.05)^{-5}
\]

{Note the change in notation \((1.05)^{-1} = 1/(1.05)\)}

\[
PV = 4.76 + 4.53 + 4.32 + 4.11 + 3.92 + 78.35 = 100
\]

Thus, if the values in parenthesis representing the reinvestment rate of the coupons received had been anything other than 1.05 the yield to maturity would not have been 5%. Yield to maturity is thus a very dangerous method of evaluating the prospective return on an investment as it is almost certain to be wrong. Clearly, if there is an expectation that rates will change, then the bond will have a different future value and a different expected return. This should cause its price to be different.

Let's look at the FV calculation again:
The total interest return is 127.6282 - 100 = 27.6282. Out of that amount 25 is represented by the coupon payments and 2.62 is the interest on interest due to the reinvestment of the coupons. This is about 10% of the total. This is the amount that is at issue with respect to reinvestment rate risk. For a 5% 5 year par bond this is no big deal. If none of the coupons had been reinvested the yield would have been \((125/100)^{20} - 1 = .0456\), about half a percentage point lower.

The importance of reinvestment risk should then be related to the size of the "interest on interest" component of the "compound" or "total" return on the bond. If we think about this, it seems reasonable that the interest on interest component should be larger when the coupon payments are larger. It should also be larger the longer the life or maturity of the bond. It should also also be related to the price paid for the bond relative to its par value. For example, if we buy the bond at a discount (i.e. a price below 100) then the bond probably has lower coupons (interest rates have risen since the bond was issued). The discount represents a capital gain which will be received at maturity that is independent of any subsequent changes in interest rates. Discount bonds should thus have lower reinvestment rate risk, and premium bonds a higher risk.

For a 10% coupon bond at par, the FV would be:

\[
FV = 14.64+13.31+12.10+11.00+10+100=161.05
\]

and the interest portion is 161.05-100= 61.05 and the interest on interest \(61.05-(10\times5) = 11.05\) which is about 18% of the total interest return.

If we compare this with a 10% bond with a 10 year maturity we have

\[
\]

Accumulated interest is $159.37, of which $59.37 is interest on interest or 37%. These percentage figures would of course have been larger had we applied semi-annual discounting.

How could we avoid this problem of reinvestment rate risk? We already know the answer. Buy a zero coupon bond which by definition pays its stated rate of interest over its whole life and thus has no reinvestment rate risk. If you bought a five-year bond today for 78.35 repaying 100 in five years you would earn \((100/78.35)^{20} - 1 = .05\) for certain, every year for five years, no matter what happened to interest rates in the meanwhile. No risk, no bother.

Thus, for coupon bonds, the longer the maturity, the greater the risk that the realised return on the bond over its life will be different from its yield to maturity. We might be tempted to think that this provides a justification for longer maturity bonds paying higher rates of interest. But, if rates are as likely to rise as to fall, then the realised return may just as likely turn out to be higher than the stated yield to maturity. If the rate of interest rose to 10% just after a 5-year 5% bond had been purchased, and its coupons were successfully reinvested at 10% then the total return would be:
FV = 7.32 + 6.66 + 6.05 + 5.5 + 5 + 100 = 130.53 for a 5.47% realised rate of return.

However, this does suggest that if interest rates are expected to change some bonds will be more favourably affected than others. For example, a high coupon bond with more reinvestment interest will benefit more from a rise in rates, while it will be hurt more by a fall in rates. This should also extend to a premium bond. Does it also extend to a longer maturity bond? It would seem so, for longer maturities will have more reinvestment interest and thus benefit more from a rise in interest rates. Usually, higher interest rates are considered a bad thing for bond investors (especially after the bond market crash of 1994!). What is wrong here?

There is one (only one?) other thing we have to keep in mind. While the realised rate of return on the 5-year 5% bond went up to 5.47% because its FV went up with the higher interest rate, what happened to its present value? Let's do the calculation. At this point it is useful to remember that the interest payments are equivalent to an annuity stream and that we can use the annuity formula developed above to do these bond price calculations:

\[ A_5 = \frac{1 - \frac{1}{(1+r_5)^5}}{\left(\frac{1}{1/(1+r_1)}\right)-1} = 18.95 \]

and the PV of 100 in five years is \(100(1.10)^{-5} = 62.09\) or PV =81.04. This is a fall in price of $18.96 (on the assumption that the change in the interest rate occurs immediately after purchase). The loss in capital value is far greater than the interest on interest gain of $2.90. Maybe higher interest rates are not so great after all.

But, we also have to remember that the value of the bond is a function of time to maturity. After one year it is a four-year bond and would then have a value of 84.15. After two years it is a three-year bond with a value of 87.57 and after three years 91.31. With one year remaining it is a one-year bond paying 105 in one year. Its present value is then 95.45. At maturity it will have a value of 100. Thus, at some point during the last year of its life the bond can be sold for a loss which is just equal to the additional interest income gained from the current value of the rise in reinvestment interest. This is a sort of "break-even" point that has to be passed before there is any real gain from the rise in interest rates.

Alternatively, at this "break even" point we could say that the bond had in fact earned the 5% yield promised to maturity since holding the bond beyond that point gives you more than 5% due to the additional reinvestment interest and selling it before produces a capital loss which has to be subtracted from the yield.

This relation works both ways; a fall in the reinvestment rate would have caused a rise in the present value of the bond, but a fall in interest on interest, so there is a break even point at which you just earn the 5% yield to maturity even though the rates at which the coupons could be reinvested were lower. Note that since a zero-coupon bond pays no interest, its break-even point is its maturity.
Since the time remaining to maturity of a bond falls with the passage of time and, ceteris paribus, the prices of bonds with shorter maturity is higher than those of longer maturity when the yield curve is positively sloped, this means that the value of a bond rises as its maturity declines. For example, the price of the 5% 5 year bond rises from 84.15 to 87.57 as it matures from a four year to a three year-to-maturity bond. This gives rise to what is called "riding the yield curve" investment behaviour. Investors borrow funds short term and invest them in longer-term bonds in order to earn a return equal to the difference between the coupon interest and the short-term funding, plus the rise in the bond’s value. The only risk is that rates might rise and cause a capital loss which exceeds the interest differential plus the positive impact of the decline in maturity. On the other hand, if rates are expected to fall, and longer rates are expected to fall more than short rates, then the return is even greater.

Explaining the Impact of Interest Rates on Bond Prices - Duration

This idea of a "break even" point is useful in understanding the risk involved in buying a bond when interest rates are likely to change. Obviously, the earlier the "break even" point occurs, the lower the risk of a return lower than the bond's promised yield to maturity, since that is the date at which you can sell the bond and get the expected yield to maturity return irrespective of reinvestment risk.

The amount of time to the "break even" point is called the "duration" (in difference from the maturity) of the bond. We have already noted that the duration of a zero coupon bond is its time to maturity. For other coupon bonds it will be less than the maturity. Measuring the break even point is a rather complicated mathematical problem since we have to find the point at which the function of the present value of the bond reaches the precise value which makes its divergence from purchase price equal to the current value of the change in reinvestment interest.

Fortunately, there is an easy way to find this value, again by using a replicating portfolio. We already know that a fixed-interest coupon bond may be represented at a series of zero coupon bonds. In the case of a 5- year 5% coupon bond there would be five each of a one, a two, a three, a four and a five-year dollar discount bond and one hundred five-year dollar discount bonds representing the repayment of principal. We know the duration of each of these zero bonds is their maturity. So, all we have to do is to take the weighted average of the maturities of all these bonds to find the duration of the coupon bond.
One five-year 5% coupon bond of unknown duration equals:

Five, one year duration, $1 zero bonds +
Five, two-year duration, $1 zero bonds +
Five, three-year duration, $1 zero bonds +
Five, four-year duration, $1 zero bonds +
Five, five-year duration, $1 zero bonds +

One hundred, five-year duration, $1 zero bonds. This is a total of 125 different dollar zero bonds.

What should we use for weights? Think of all 125 of the zero bonds as representing a portfolio. The contribution of each zero bond to the total value of the portfolio will be its fair value relative to the fair value of the portfolio. The weights will then be the value of each maturity class of discount bond relative to the total portfolio value.

The fair value of the portfolio is just the value of the original 5%-year coupon bond, which is selling at par or 100. The fair value of each zero bond will then be given by the very first calculation we did above, the fair value of $1 in one year when the interest rate is 5%, the fair value of $1 in two years etc., each multiplied by the size of the coupon, in this case each is $5, except for the last one which is the repayment of principal or $100.
Thus, the weight of the five one-year zero coupon bonds is \(\frac{[1/(1.05)] \times 5}{100}\). The five two-year zero bonds have a weight of \(\frac{[1/(1.05)^2] \times 5}{100}\) and so forth up to the repayment of principle which has a weight of \(\frac{[1/(1.05)^5] \times 100}{100}\). These weights are then multiplied by the duration of each class of zero bond (which by definition is equal to its maturity) to get the weighted average maturity of the portfolio of zero coupon bonds. This will give the duration of the portfolio as:

\[
\begin{align*}
5 \text{ 1-year bonds} &= \frac{[5/(1.05)]}{100} \times 1 \\
5 \text{ 2-year bonds} &= \frac{[5/(1.05)^2]}{100} \times 2 \\
5 \text{ 3-year bonds} &= \frac{[5/(1.05)^3]}{100} \times 3 \\
5 \text{ 4-year bonds} &= \frac{[5/(1.05)^4]}{100} \times 4 \\
105 \text{ 5-year bonds} &= \frac{[105/(1.05)^5]}{100} \times 5.
\end{align*}
\]

If we sum up the weights multiplied by maturity and then divide by the value of the portfolio, 100,

\[
D = \frac{\left(\frac{5}{1.05} \times 1 + \frac{5}{1.05^2} \times 2 + \frac{5}{1.05^3} \times 3 + \frac{5}{1.05^4} \times 4 + \frac{105}{1.05^5} \times 5\right)}{100}.
\]

\[
D = \frac{4.76 + 9.07 + 12.957 + 16.45 + 19.588 + 391.76}{100} = 4.545
\]

The break-even point for the bond thus occurs a little more than 6 months into the fifth year (to be more precise we would have to know the interest basis and the precise date at which the bond was purchased). Thus, after 4.545 years the bond will have achieved a realised rate of return of 5%, **no matter what the change in interest rates which takes place after the initial purchase at par**.

Now the question is, what are the factors that determine the duration of a bond? First of all, as the replicating portfolio used above makes clear, it depends on the weights applied to each of the zero bonds which make up the replicating portfolio. The weights are given by the time period and the price of the zero bond maturity category relative to the whole portfolio. Thus, the interest rate and the size of the coupon and maturity will influence the weights and thus the duration.

Holding the coupon and yield constant, duration increases with longer maturity. Holding maturity and yield constant, higher coupon bonds have lower duration because principal payments are lower. A higher coupon means the weights on the shorter term zero bonds will make a larger contribution, so that should make duration shorter. Holding maturity and coupon constant, lower yields should cause duration to rise. This is because lower interest rates will make the prices \(1/(1+r)\) of the shorter-term zero bonds higher, BUT the impact on the prices of longer term zeros will be even greater because they have higher exponents \(1/(1+r)^5\), making the impact on their weights even larger. Thus, lower yields to maturity should make duration longer.

A bond selling at a premium to par, (and thus usually a bond with higher coupons than on currently issued bonds), will lose return on principal repayment, so it should also have a shorter duration. Bonds selling at a deep discount, or with non-standard coupon streams may have an anomalous behaviour as maturity increases. At longer maturities they increase in duration, but after a certain point the majority of the bond's
value, which is in the capital gain at repayment of principal, is so far in the future that its contribution approaches zero and the duration falls.

In general (other things equal):
Duration increases with: Lower Coupons - Higher principal repayment
Lower yields to maturity
Longer maturity (par and premium and most discount bonds).

If we remember that the present value of a bond is made up of three components: principal, coupon interest and interest on coupon interest, a rise in interest rates has positive effect on interest on coupon interest, and leaves the absolute values of principal and coupon payments unchanged. However, it does have a negative impact on the last two elements when they are discounted to the present.

We have defined duration as the point in the life of the bond at which these two opposing forces balance each other. But, what can duration tell us about today, in the present. Any change in the current interest rate will change the present discounted value of the unchanged principal and coupon interest, as well as the present discounted value of the higher or lower interest on coupon interest payments. This means that a change in the rate of interest will have a differential impact on bonds with the same prices, but with different durations. If you are a bond trader, this will be important in setting the buying and selling prices on the bonds in your portfolio. You would want to sell out those bonds whose prices will fall most, so you might offer more attractive (lower) prices.

Duration would also be useful if you are a portfolio manager and are asked to provide a guaranteed rate of return over a fixed period, called the investment horizon. If you invest in bonds which have a duration equal to your investment horizon, and hold them until that date, you are certain of the total return you will report on the portfolio, irrespective of what happens to interest rates over the investment period. This is called a "dedicated" portfolio, since it is dedicated to a particular return objective and future date.

Duration also forms the basis for calculating the price response of a bond to a change in yields. Write the price function of the five year bond:

\[ PV = 5(1+r)^{-1} + 5(1+r)^{-2} + 5(1+r)^{-3} + 5(1+r)^{-4} + 5(1+r)^{-5} + 100(1+r)^{-5} \]

To find the response of the price of the bond to a change in the rate of interest take the derivative of the price function with respect to the interest factor, 1+r:

\[
\frac{dPV}{d(1+r)} = -1*[5(1+r)^{-1}] -2*[5(1+r)^{-2}] -3*[5(1+r)^{-3}] -4*[5(1+r)^{-4}] -5*[5(1+r)^{-5}] -5*[100(1+r)^{-5}].
\]

Now, multiply both sides by (1+r)

\[
[dPV/d(1+r)](1+r)= -1*[5(1+r)^{-1}] -2*[5(1+r)^{-2}] -3*[5(1+r)^{-3}] -4*[5(1+r)^{-4}] -5*[5(1+r)^{-5}] -5*[100(1+r)^{-5}].
\]

The negative impact of an increase in the interest rate on bond price is now clear from the minus sign.

Compare the first term of this relation \{-1*[5(1+r)^{-1}]\} with the first term \{5/(1.05)*1\}/100 of the weighted
average portfolio we used above to calculate duration. Recalling that $1/(1.05)$ may be written $(1.05)^{-1}$, it is clear that the two terms are identical if we multiply the latter by 100, which is the present value of the bond. Since all the individual terms of both relations are identical, we can write

$$[dPV/d(1+r)]*(1+r) = -D * PV \text{ where } D = \text{duration.}$$

By rearranging terms we can find

$$dPV/PV = -D/(1+r) * d(1+r) = -D/(1+r) * 1 * (dr) \text{ and}$$

$$dPV = -D/(1+r) * PV * dr$$

-D/(1+r) is called modified duration. This relation allows us to calculate the change in the absolute $ price of the bond when r changes by dr. Say interest rates change by one basis point (bp=1 hundredth of a percent) to .0501.

$$dPV = -4.545/(1.05) * 100 * .0001 = -.0432857.$$  

This is often called the basis point value (BPV) used in the evaluation of bond futures. It is the change in value represented by a one bp change in yield. Had we left the relation as dPV/PV, then instead of an absolute value, we would have found the percentage decline in the bond price.

The price of the bond will fall from 100 to 99.9567, or a little over 4 cents. On a million dollar bond portfolio, that works out to over $400 per basis point change. Changes of 20 or 30 basis points are not uncommon, which produces a change in portfolio value of $10,000.

Modified duration thus provides a measure of the sensitivity of bond prices to a change in interest rates. The larger the value of modified duration, the larger the change in price for a 1 bp change in interest rates, and thus the higher the risk of a change in capital value associated with the purchase of the bond.

Two warnings: since the measure of duration is a function of interest rates, it will change when interest rates change. It is not constant. Nor is its response to changes in rates predictable because it depends on changes in the spot yield curve - more on this below. (You will see similar relationships, called deltas, used in the analysis of changes in prices of futures and options contracts. The change in the values of the deltas are called gamma.) As a result, the use of modified duration to measure price movements only works for infinitesimal calculus type "small" changes in yield, because for large changes the measure of duration will itself be changing. To take this into account we need to take an additional derivative of the price function and measure what is called "convexity". (This is the gamma part. It is stuff for real experts.)

Yield Curves and Yield Differentials: Term Structure of Interest Rates

The idea of duration as a break even point and a measure of bond price sensitivity gives us some help
in explaining the relation between the maturity of a bond and its rate of return. There is a clear relation between a bond’s duration and the change in its present value when interest rates change. The longer the duration, the larger the change in price or, to put it another way, the longer you have to wait to get to the break-even point at which the impact of a change in interest rates on total yield is eliminated. If there is an expectation that interest rates will rise, then for a given accepted variation in portfolio value, there would be a preference for bonds of shorter duration. This will drive up their prices relative to bonds of longer duration and cause their yields to fall relative to bonds of longer duration. Yields of the longer duration bonds thus have to rise sufficiently to offset the higher risk of a loss in their capital value.

This allows us to explain what is usually observed in the market, that is, a rising curve of yields of bonds ranked according to maturity (and thus roughly according to duration, since as we have seen there is a positive relation between maturity and duration).

This is referred to as the "expectations" theory of the yield curve. Consider an investor with a two-year investment horizon, i.e. with the objective to maximise wealth at the end of two years. There are one-year and two-year bonds available in the market and the current interest rate is 5%. The choices available to the investor are thus to buy the two-year bond, or to buy a one-year bond and then buy a second one-year bond in the second year. If expectations of future interest rates are stable, i.e. interest rates are expected to remain fixed at 5%, then there is no difference between the two possibilities. However, if interest rates are expected to rise during the first year, to say 10%, then conditions change. The total return for the two-year bond is $5 in the first year and $5+0.25 in the second year. The expected return from the two one-year bonds is $5 in the first year and $10+0.50 in the second year. Thus, no one will be willing to buy two-year bonds at par, their price will fall and their yield will rise. Likewise, the preference for one-year bonds will cause their price to rise and their yields to fall. The end of the story is lower yields and higher prices for one-year bonds than for two-year bonds.

The expected annual return over two years earned by investing consecutively in one-year bonds is \((\frac{115.5}{100})^2 - 1 = 7.47\%\). To compete with the one-year bonds the price of the two-year bond has to fall to 95.56% to give a yield to maturity of 7.47%. We thus have a yield curve with one-year bonds at 5% and two-year bonds at 7.47%. Longer term instruments now have higher yields and the yield curve is upward sloping. This is a positive yield differential of long over short bonds of 247 (747-500) basis points (bp).

If we calculate the one-year forward rate implicit in the repriced two-year bond we would have to find the present value of the $5 coupon at 5% which is $4.76. This gives us a two-year zero with a current price of 95.56-4.76 = 90.798 or 7.53%. The one-year rate one year ahead is then constructed by creating a portfolio in which you sell the one-year bond and buy the two-year zero. This gives an income of 95.238 from selling the one-year zero and a cost of 90.798 to buy the two-year or $4.44 of net income today. This will be worth $4.66
in one year at 5%. One year from now the bond sold has to be redeemed at par or 100, less the 4.66 earned for a net cost of 95.34 for the 105 that will be received in another year from the two-year zero bought. This gives a return of 10%, which is the one-year forward rate for one year.

We thus have a coupon yield curve of:

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<tr>
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<th>coupon bond</th>
<th>zero bond</th>
<th>1 year forward rate</th>
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<tbody>
<tr>
<td>1 yr</td>
<td>5.0</td>
<td>5.0</td>
<td>10%</td>
</tr>
<tr>
<td>2 yr</td>
<td>7.47</td>
<td>7.53</td>
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** We need the 3-year rates to be able to bootstrap the 2-year forward rate.

When the yield curve is positively sloped the zero coupon yield curve is thus above the coupon yield curve, and the forward curve is above the zero curve.

Note that this explanation of the slope of the yield curve is reversible -- an expectation of a fall in yields will produce the opposite result and what is called an "inverted" yield curve with a negative yield differential: short rates higher than long rates. This has led many experts to the idea that the yield differentials between long and short-term bond rates might conceal information about the future evolution of interest rates or of the performance of the economy or of the stock market (to name but a few that have been proposed with varying degrees of predictive success).

Note that the explanation of the positive slope of the yield curve rests on the expectation of the one-year rate in one year, and in two years, and in three years, etc. Indeed, we have reasoned on the basis that the expectation is perfectly certain. This means that the one-year forward rate should be the best estimate of that rate. In fact, it seldom turns out to be a good predictor, usually over-predicting the rise in future rates. Market experience thus suggests that there is something else influencing forward rates besides simple expectations. The factor which is most often suggested is that longer term instruments require a "liquidity premium" to convince investors to buy them.

This time, consider someone who wants to borrow for two years at the lowest rate possible. Clearly, if interest rates are expected to rise, it is better to borrow for two years. We thus have a conflict -- investors want to lend short and borrowers want to borrow long. If there is an excess of two-year bonds coming into the market, then a premium has to be offered to buyers to offset their fear of a rise in rates. This is the liquidity premium, and it is linked to the risk of a change in rates over the holding period of the bond and the imbalance between buyers and sellers for a given set of future expectations. The longer the maturity, the higher the risk premium required, although it usually is presumed to be a decreasing function of the duration of a bond.

A variant of the liquidity premium is the segmentation or preferred habitat theory. This says that some investors have preferences for certain maturities of bonds, and premia are necessary to convince investors to
move to other segments of the yield curve when there are imbalances in demand and supply in particular segments. This would cause forward rates to depart from the pure expectations explanation. For example, banks may prefer shorter maturities, while insurance companies may prefer longer maturities.

In any event, casual empiricism (looking at the recent data) suggests that most of the time the yield differential between long and short-term bonds is positive. So that this has come to be accepted as normal. The yield curves for coupon bonds, for zero bonds, and for forward rates are all upward sloping. However, this has not always been the case. Under the gold standard inverse yield curves prevailed whenever the structure of rates was high and vice versa. This has led some to conclude that the slope and behaviour of the yield curve is influenced by the monetary standard. It was different under the gold standard of the 1880s than under floating exchange rates in the 1980s.

Whatever is considered as the "normal" shape of the curve, what is not normal is a flat yield curve, i.e. yields constant over the maturity or duration spectrum. There is a difference which is equivalent to that between Euclidian and non-Euclidian geometry associated with the passage from flat to sloping yield curves. Calculations become much more complicated.

First, as we saw above in our simple examples, in a flat curve world, there is no difference between yield to maturity, spot yields and forward yields. If the curve is flat then the two-year annualised spot rate is the same as the one-year rate and both are the same as yield to maturity. Further, forward rates are also the same.

Second, in a flat world changes in interest rates will bring about a uniform shift in the entire curve. Outside of this world a change in rates need not change all rates uniformly or proportionately, so that we have to be careful about the influence of the change in the slope of the curve on bonds with similar prices but different income streams. We thus have to be concerned about the impact of a shift in the curve on its slope (the “twist”) and on its curvature (whether it is “humped”).

In practice what does this imply? For a start, recall our calculation of duration as a portfolio of zero's. We used the same 5% rate for each of the five different zero maturities. This is wrong! We should have used the spot or zero rate corresponding to each maturity. When we introduced a change in the interest rate to see the impact on duration, we presumed a uniform shift in the yield curve. This was wrong. Each of the spot or zero rates will change in different proportions. Thus, when we value a bond we should evaluate each coupon with respect to its appropriate spot rate for the maturity of each cash flow. Thus, the present value of a five-year, five percent coupon bond's income flows should be:

\[
P V = C_1 \cdot (1+r_1)^{-1} + C_2 \cdot [(1+r_1)(1+r_2)]^{-1} + C_3 \cdot [(1+r_1)(1+r_2)(1+r_3)]^{-1} + C_4 \cdot [(1+r_1)(1+r_2)(1+r_3)(1+r_4)]^{-1} + (C_5 + F) \cdot [(1+r_1)(1+r_2)(1+r_3)(1+r_4)(1+r_5)]^{-1}
\]
where $C_n$ are the coupon payments in each of the $n = 5$ years of the bond's life and $r_n$ are the spot rates for each of the $n = 5$ years. Any change in interest rates can then be traced through to the individual spot rate and its impact on each individual coupon.

All of this means that the fair value of any instrument should be calculated using the appropriate spot rates in order to be able to make meaningful comparisons between different bonds. This is because your return on holding the bond, or holding period return, will be determined by three factors, the size of the coupon, the discount or premium, and the risk as measured by the sensitivity of price to changes in interest rates. Say you have two bonds with 5 years remaining to maturity, one was issued when rates were low and pays a $5 coupon, the other was issued when rates were high, and pays a $10 coupon. The first pays 8.78% yield to maturity and the second 8.62%. The five-year spot rate is 9% and their prices are 85.21 and 105.43 respectively. How can you compare them? There is no benchmark yield to maturity for 5 year bonds.

First, you can decompose them into six zero coupon bonds. You can calculate the fair value of each by using the appropriate spot interest rate. Then you can calculate the duration of each, again using the appropriate spot rates. Then you can calculate the impact of a change in rates on their values, including a change in the spot yield curve. This will depend on your expectations of changes in rates. In general, the $10 coupon bond has shorter duration, so that would be one reason for its higher price and lower yield to maturity in an environment in which yield curves are upward sloping and rates are expected to rise.
Annex: Payment Frequency and Interest Calculations

Up to this point we have been working with a time unit equal to a year; all the interest rates were normalised on an annual basis. For example, the return on the purchase of $1 over two years, i.e. 
\[
\frac{1}{1.7} - 1 = 42.85\%
\]
which is equivalent to a two-year zero interest rate bond, was transformed into an annual rate by taking the annual geometric mean value to represent its return. We could have left it as a two-year rate. But, this would have meant that the return on the one-year bond would have had to have been adjusted by squaring it to give 37.98% as its biannual or two-year yield, so we would not have been much ahead. We will always have to be converting returns to rates on some normalised basis, so it may as well be a year. But, this means that we have to do another set of adjustments for instruments of less than a year, or for bonds that pay their interest more frequently than yearly.

The most common adjustment of this sort is for US bonds which pay interest semi-annually. Instead of paying $1 in coupon interest per year for a return of 17.6471% per year, the bond in our example would pay coupon interest of $0.50 every six months. You can also think of the bond as paying half the annual interest or 8.82355% per six month period.

Is this a good thing? Well, it might be, and then again it might not be. What it does mean is that you get half of your money six-months earlier than with an annual coupon. Remember reinvestment risk on coupons? If rates have gone down, you are worse off, if they have gone up you are better off. All you know is that you get to reinvest the money earlier.

What it does mean is that at the end of the year instead of getting $1 in coupon interest you will get $1 plus whatever you have managed to earn on the $0.50 over the last half of the year. If 17.6471% is the one-year interest rate, then we earn 
\[
\frac{1}{2} \times (1 + 0.0882355)
\]
Note that we have divided the coupon by 2 since that is the amount invested for the second half of the year, and we have also divided the rate of interest by 2, because we only have it invested for half the year. If the interest rate remains unchanged we will get 
\[
0.5 \times [1.0882355] = 0.544
\]
plus the second $0.50 payment, for a total of $1.044, instead of one dollar at the end of the year. We can thus conclude that the bond which pays semi-annually should be worth more, ceteris paribus. What is the fair value of the bond with a semi-annual coupon compared to an annual coupon bond?

We could proceed by calculating the fair price of $0.50 in six months. This would be 
\[
1/1.0882355 
\]
$.50 = 45.9459 cents. We already know the fair price of $1.00 in 12 months is .85. But now there is an alternative way of getting the fair price. It is the same procedure that we used for the two-year bond above, but now it is for two periods of six months. What we have to do is to find the fair price of a portfolio composed of a six-month $0.50 discount bond and a one-year $0.50 discount bond. $0.50 in six-months time has a fair value of 45.9459 cents. To get 50 cents in one year we buy 45.9459 cents for delivery in six months
and reinvest it for six months to yield $0.50. This will cost $0.4221. Thus, 42.22 cents for six months gives 45.9345 cents, which reinvested for a second six months gives 50 cents. This gives $0.50 + 42.22 = 88.17 as the fair price of two payments of 50 cents at a six months payment frequency.

We know that we should be able to get the same result by doing the present value calculation and discounting the 50 cent payments: $0.50/1.0882355 + $0.50/(1.082355)^2 = .459459 + 42.22 = 88.17, which is the present value of the first year's coupon payments on the bond. We can generalise this to say that when payments occur more frequently than once per year we have to divide the interest rate by the number of annual payments (in this case .176741/2), divide the annual coupon by the number of payments per year [in this case 1.00/2 = .50] and multiply the exponent on the interest factor by the number of payments per year [in this case $1/(1.179471)^{1/2}$ for the value of the first annual coupon payment becomes $1/(1.0882355)^{1/2}$] for a bond paying interest semi-annually. Finally, we have to add the new coupon payments [in this case the payment after six months which is $.50/(1.0882355)].

Thus if C is the coupon and r is the interest rate our calculation of the fair value of the first year's payments goes from $C/n * 1/[1+(r/n)]^n$ where n = 1 to \{C/2 * [(1/1+(r/2))^1] + C/2 * [(1/(1+(r/2))^2)]\} for n = 2 periods. If the payment had been quarterly (i.e. every three months), then we would have n = 4 and

\{C/4 * [(1/1+(r/4))^1] + C/4 * [(1/(1+(r/4))^2)]\} + \{C/4 * [(1/1+(r/4))^3] + C/4 * [(1/(1+(r/4))^4)]\}.

When this is discussed with reference to the future value, FV, of a present sum it is called the "frequency" of compounding, i.e. how frequently is interest added to the initial sum. Annual compounding adds interest once every year: $1.0 * (1+r/n)^n$ and n = 1, semi-annual compounding adds interest twice a year $1.0*(1+r/2)*(1+r/2) = 1.0^2*(1+r/2)^2$, and so forth until we get to daily (n = 365) compounding. How far can we go? The limit case would be every instant of time, or "continuous" compounding. Although your banker can't write your interest into his account books that fast, he can still calculate what your account would be at discrete intervals as the value of n approaches infinity.

We have already noted that the accumulation of interest may be represented by a geometric progression [i.e. we multiply \((1+r)(1+r) = (1+r)^2\) rather than add \(1+(r+r) = 1+2r\)]. The exponent represents the number of payment periods which we have called n, times the number of normalised time periods, call them t. They represent years if we normalise on an annual basis. Accumulated interest, AI, is then the initial sum, which we called present value, multiplied by the interest factor, adjusted for n and t or \(AI = PV[1+(r/n)]^{nt}\). For annual payments, n = 1. For annual rates, t is implicitly measured in years. But, payments may be made more frequently and interest can be calculated on any basis we like.

Define nt = n/r^t. Write \(PV\{[(1+(1/m)^{m})^{nt}\]. Now substitute m for n/r and rewrite this relation as:

\(PV\{[(1+(1/m)^{m})^{nt}\].

Concentrate on \([1+(1/m)]^{m}\). Think of a large number, say m = 10,000 and do the calculation
The answer is 2.718145927. Now think of a very large number, \( m = 1,000,000,000 \).

The answer is 2.71828. My calculator overflows after this. A mathematician will tell you that this is a convergent series, and that if you could keep going, increasing \( m \) up to an infinitely large number, you would always come up with the same number, 2.71828. This happens to be the base of the system of natural logarithms. Note that as \( m \) goes to infinity, \( n \), the frequency of compounding, which is \( n = rm \), will also approach infinity. Infinitely frequent compounding is then the formal definition of continuous compounding of interest.

This means that the banker doesn't have to stay awake all night doing interest calculations. All he does is to stick the number 2.71828 in place of \((1+(1/m))^m\) in the accumulated interest formula given above and solve for \( AI \):

\[
AI = PV \times (2.71828)^t = PV \times (e)^t.
\]

If this is for one year, \( t = 1 \), so a $1 deposit, compounded continuously at 17.6471% would have a credit balance of 2.71828\(^{1.176471} = $1.193 \) at the end of the first year. We can thus compare the various possibilities:

- Continuous = $1.193
- Daily = $1.00 \times (1 + .176471/365)^{365} = $1.19295
- Monthly = $1.00 \times (1 + .176471/12)^{12} = $1.1914
- Quarterly = $1.00 \times (1 + .176471/4)^{4} = $1.1884
- SemiAnnual = $1.00 \times (1 + .176471/2)^{2} = $1.1842

The difference between continuous and annual compounding is only 1.7 cents, not a big deal, but in percentage terms that is 10%!

We could also have calculated the continuous compounding rate for a longer or shorter period than a year, by adjusting the exponent \( rt \) by a value of \( t \) greater or lesser than 1 year. A monthly rate would give \( e^{r/12} \).

It is also possible to apply more frequent compounding to annual coupon flows. For example, a two-year bond with annual payments may be compounded on a semi-annual basis by using semi-annual discount factors:

\[
PV = C/[1+(r/2)]^2 + (C+F)/[1+(r/2)]^4
\]

Note that \([1+(r/2)]^2 \) replaces \((1+r)^1\) and \([1+(r/2)]^4 \) replaces \((1+r)^2\) of the original formula.

Alternatively, semi-annual coupons may be discounted on an annual basis:

\[
PV = [C/2]/[1+r]^5 + [C/2]/[1+r]^3 + [C/2]/[1+r]^1 + [(C/2)+F]/[1+r]^2
\]

In this case \([1+r]^5 \) replaces \([1+(r/2)]\) and \([C/2]/[1+r]^3 \) replaces \([1+(r/2)]^2\) of the original formula.

Finally, we can find equivalences between annual yields \((r_{12})\) and semi-annual yields on an annual basis \((r_6)\) as
follows:

\[ 1 + r_{12} = \left(1 + \left(\frac{r_6}{2}\right)\right)^2. \]

Taking the square root of both sides gives:

\[ (1 + r_{12})^{\frac{1}{2}} = 1 + \left(\frac{r_6}{2}\right). \]

Subtracting 1 from both sides gives:

\[ (1 + r_{12})^{\frac{1}{2}} - 1 = \left(\frac{r_6}{2}\right). \]

Multiplying both sides by 2 gives:

\[ \left((1 + r_{12})^{\frac{1}{2}} - 1\right)\times 2 = r_6. \]

This relation can be used to convert annual to semi-annual compounding.

Reversing the procedure and solving for \( r_{12} \) gives:

\[ r_{12} = \left[1 + \left(\frac{r_6}{2}\right)\right]^2 - 1. \]

If we do the calculation for \( r_{12} = 5\% \) this gives \( r_6 = 4.9390154\% \) semi-annual. This means that you get the same absolute interest return from a 5\% annual compounding as from a 4.939\% semi-annual compounding.

\$1[1+(.04939/2)]\ gives 1.02469 after six months and 1.05 after one year.

This can also be applied for quarterly compounding

\[ r_3 = \left((1 + r_{12})^{0.25} - 1\right)\times 4. \]

Obviously, this also applies to the discount factors, \( 1/r_{12} \), etc. as well as to yields to maturity on bonds, i.e. we can use these formulae to compare yields to maturity quoted on bonds with different coupon payment flows and thus different compounding of interest received.

While we are on the subject of time periods, it is useful to be alert to a problem that came up when we looked at the determination of the rate of interest on discount instruments such as Treasury bills. Although we talk about years and months, there are no standard years and months. Remember the jingle, 30 days in September, April, June and November, all the rest have 31 except February which has 28, and in leap year 29. Which also says that some months are bigger than others, (and should produce more interest) and although every year has 12 months, some have 365 and some 366 days.

There is no clear solution to this problem and some value calculations use the actual number of days, some use "notional" months of 30 days, which produces a "notional year" of 360 days. Most US bond yields are calculated on a 360 day year, while the UK uses a 365 day year -- so be careful when you compare UK bond yields with US bond yields. Also, even within one country, such as the US, different instruments will use different conventions so that you have to be careful when comparing interest rates.

In general, we can define **simple interest** as the result of the calculation of Principle \( \times \) interest rate \( \times \) time to maturity or:

\[ P \times r \times t = \text{Interest}. \]
The time, t, used to calculate interest can be notional, e.g. a month of 30 days per year of 360 days (written 30/360). Or it can be the actual number of days per notional year (written: actual/360), or be on an actual number of days per actual number of days in the year (written: actual/actual) basis. Every instrument has its own calendar.

The interest rate calculation will be on an "interest bearing" or a "discount" basis. For example, a 10% one-year interest-bearing instrument pays 10 cents on every dollar of the instrument purchased if interest is calculated on an actual/actual (365/365) basis (and it is not leap year). If it is quoted on an actual/360 basis then the interest is 10*(365/360) = 10.14 cents.

A 10% discount instrument on an actual/actual basis would have a purchase price of 90 cents per $1.00 of face value or a discount of 10% on the par or face value or maturity value of 100. On an actual/360 basis the discount would be 10*(365/360)=10.14 cents and the price, 100-Discount = 89.86 cents. The interest accrued would then be 100-89.86 = 10.14 cents and the rate of return on a simple interest basis would be 10.14/89.86 = .1128 or 11.28%. It is thus clear that discount basis instruments cannot be directly compared to interest basis instruments without conversion of the discount rate to the appropriate rate of return.

The other factor which affects interest earned will be frequency of compounding, and we have already seen how we can convert different frequencies. The basic problems in comparison come from conventions which have been developed by traders in specific markets. The first is the fact that US government bonds pay interest semi-annually, and therefore have to be converted to an annual payment basis to be compared with other bonds. By market convention this is done simply by multiplying the semi-annual yield to maturity by two. Thus, the yield to maturity for a two-year bond paying semi-annual coupons is found by solving:

\[ PV - \{(C/2)/(1+j_{6}) + (C/2)/(1+j_{6})^2 + (C/2)/(1+j_{6})^3 + (F+C/2)/(1+j_{6})^4\} = 0 \]

for the value of \( j_{6} \).

Then the bond equivalent yield or coupon equivalent yield is simply produced by rule of thumb as \([2 \times (j_{6})]\) rather than the "correct" equivalent calculated on a continuous compounding basis, which would be \((1 + j_{6})^2 - 1\).

Another problem comes from the basic discount instrument sold in the money market, the US Treasury bill. T bills are sold via a competitive bid auction on a "bank discount basis". The annualised yield on a bank discount basis for a one year (12 month) bill is defined as \([\frac{FV-PV}{FV}] * 360/364\). Note that the bank discount basis is 1-(PV/FV) rather than the (FV/PV)-1 normally used to calculate the rate of return. For purposes of definition, note that FV-PV=D the discount from the face value and the bank discount basis return is the discount as a proportion of face value.

There are two corrections that have to be applied to make this into an annualised yield. First, the discount (D/F) is on a 360 day basis. Second, the maturity of the bill is 52 weeks, which is 364 days. Thus to
convert the notional 360 day rate to an actual 365 day rate basis requires multiplying by 360/365. Then the actual 364 day holding period has to be adjusted to an actual 365 day year by multiplying by 365/364. The product of these two corrections \{(360/365)*(365/364)\} produces the 360/364 correction to the bank discount basis, which gives the annual equivalent rate of the formula. Remember, we said that even annual rate bases raised complications.

Hopefully, an example will make things clearer. For a 52 week Treasury bill with a 10% issue discount the correction for the 360 day basis (365/360) transforms the discount to .10*365/360 = .1011. The Discount on a notional face value of 100 is then .1011*100 = 10.11. The Price or PV is then 100-10.11 = 89.89. The return from buying the bill at issue and holding to maturity 364 days later is then 10.11/89.89 = 11.25%. This has to be converted to a 365 day basis so that it can be compared with a coupon bond. Multiplying by 365/364 gives 11.28% as the bond equivalent yield for the 12 month bill.

If we define the corrected discount rate on the bill, d, as (1-PV/FV)*360/364, then the bond equivalent yield for the bill is given as (365*d)/[360-(d*364)].

For the present case d = .09999, and the bond equivalent yield is given as (365*.0999)/360-(.0999*364) = 11.27935%.

An investor buying a par coupon bond paying a fixed coupon of $11.28 semi-annually would receive two payments of 5.64. The yield to maturity of this bond would be .0564 * 2 = 11.28%. Thus the effective rate of return earned from buying the coupon bond and the discount bill are equivalent at 11.28%.

This can be generalised to bills of tenor (this is a fancy name for its time to maturity) less than 52 weeks as d = (1-PV/FV)* (360/t) and bond equivalent yield = (365*t)/360-(d*t) where t is the time remaining to maturity. A rose by any other name ...

Let's apply these complications to our present value calculation. On a continuous compounding interest basis our original calculation of the value of $1 in a year would become $1* e^{-0.176471} = .83822. As noted above we have substituted e for 2.712871. Since its logarithmic value is 1, this means that if you remember how logs work you can do this calculation as log AI= -0.176471*log 2.71828 = -0.176471 * 1. So, if you find the antilog of -0.176471 with your computer or with log tables you will find the result 0.83822 directly.

Now, if we had done the calculation for the dollar payment in two years, the value of the second year $1 payment would have been $1* e^{-0.176471*2} = 0.7026. This gives the value of the two payments as $1.54.