A one-commodity model

In this chapter the mathematical properties of a model in which only one commodity is produced by means of itself and homogeneous labor will be studied. While this model is exceedingly simple from the mathematical point of view, it is difficult from the conceptual one. The economic system contemplated is a highly abstract one in which it is assumed that only one commodity is produced; it is assumed that this commodity is produced by means of itself and homogeneous labor. The only relative price dealt with is the real wage rate. This model is introduced for illustrative purposes only. However, several of the concepts developed in this chapter and some of the results derived carry over, with some modifications, to the more general models dealt with in the subsequent chapters.

The structure of the present chapter is as follows. In Section 1 the technical conditions of production are described and discussed. Section 2 introduces the concept of viability. Section 3 deals with growth and consumption opportunities, Section 4 with the distribution of the product between workers and capital owners. Two alternative views concerning the connection between the growth–consumption and the wages–profits aspects of the system under consideration are discussed in Section 5, in which saving and investment behavior is introduced. Section 6 is devoted to the problem of the choice of technique. Section 7 deals with alternative conceptualizations of the relationship between distribution and growth. Section 8 provides some historical notes, Section 9 contains some exercises.

1 In this chapter as well as in Chapters 3 to 9 it will be assumed throughout that natural resources are non-scarce. Therefore the problem of rent paid to the proprietors of these resources does not arise.
1. Technology

Consider an economy which produces only "corn." At the beginning of the production period, which is taken to be equal to a year, corn is sowed by laborers and at the end of the period the crop is harvested. Hence corn is the only produced means of production (seed-corn) and the only means of subsistence (food) to support the population. The production technique can be represented, in abstract terms, as follows:

\[ a^* \text{ bushels of corn} \oplus l^* \text{ hours of labor} \rightarrow b^* \text{ bushels of corn}. \]

The symbol "\( \rightarrow \)" stands for the "black box" in which laborers working for \( l^* \) hours at a given intensity of work are transforming \( a^* \) bushels of corn into \( b^* \) bushels of corn during the yearly production cycle. The symbol "\( \oplus \)" in the present context indicates that both inputs, that is, corn and labor, are required to produce the given output of corn. The period of production separates the moment when the sowing starts from the moment when the harvesting is accomplished.

It is convenient to set \( b^* \) equal to unity. This can be done in two ways. The first consists in changing the physical unit in terms of which corn is measured from bushels to what may be called "\( b^*\)-units." We then have:

\[ \frac{a^*}{b^*} \text{-units of corn} \oplus l^* \text{ hours of labor} \rightarrow 1 \text{ \( b^*\)-unit of corn}. \]

Obviously, the old and the new relationship describe exactly the same economy, even though they use different units of account. Moreover, no assumption about returns to scale is required in the switch from bushels to \( b^*\)-units.

The second possibility assumes constant returns to scale in the economy under consideration. With this assumption the production technique can be represented as:

\[ \frac{a^*}{b^*} \text{ bu. of corn} \oplus l^* \text{ hours of labor} \rightarrow 1 \text{ bu. of corn}. \]

While the first alternative has the advantage of not requiring any assumption about returns, it cannot be used if the quantities produced change. The second alternative, however, requires such an assumption, but allows variations in the quantity produced. In this book, constant returns to scale will be assumed whenever it is necessary for analytical convenience.

To simplify the notation, the two previous representations of the technology are rewritten in Table 2.1. In the following discussion the reader will adopt whichever interpretation is applicable to the particular
case dealt with \( a = \frac{a^*}{b^*} \) in both cases; and \( l = l^* \) in the first case or \( l = \frac{l^*}{b^*} \) in the second).

In the present context, in which there is only one kind of capital good, corn, and only one kind of primary input in the production technique, homogeneous labor, the capital intensity or capital-labor ratio of the technology used can easily be ascertained: it is given by \( a/l \). Similarly, the (net) output-labor ratio, or labor productivity, and the net output-capital ratio are given by \((1 - a)/l\) and \((1 - a)/a\), respectively.

2. Viability

An economy is said to be viable if the technology at its disposal allows it to reproduce itself (assuming labor to be available at no cost). Hence, the notion of viability used in the present book refers to the technical conditions of production only, whereas the provision of the means of subsistence needed in support of the population is not a part of it. Therefore, the viability of the system as it is defined here is a necessary but not sufficient condition of the survival of the economy. Keeping this in mind, an economy whose technology is described in Table 2.1 is viable if and only if

\[ a \leq 1. \]

If \( a = 1 \), the economy is just viable since it can reproduce itself, but in order to do so the entire harvest has to be sowed. If \( a < 1 \), the economy produces a surplus over what is needed as seed-corn and the harvest can be divided in two parts: \( a \) and \((1 - a)\). The first part can be used for reproduction purposes, while the second may be devoted to other purposes.

3. Growth and consumption

The surplus can be used either for consumption or for accumulation. If the surplus or parts of it are saved and invested the economy will grow. Let \( g \) denote the growth rate and \( c \) the consumption per unit of labor employed, or, for short, consumption per head or per capita, then \( g \) and \( c \) must satisfy the following equation (assuming constant returns to scale):

\[ (1 + g)a + cl = 1, \]

(2.1)
that is, one unit of output is used in part for consumption \((c)\), in part for reproduction \((a)\), and in part for growth \((ga)\). Equation (2.1) implies that

\[
c = \frac{1 - (1 + g)a}{l}.
\]

The maximum growth rate, \(G\), is the growth rate corresponding to \(c = 0\), that is, when the whole surplus is saved and invested:

\[
G = \frac{1 - a}{a}.
\]

Obviously, \(G\) equals the net output per unit of capital. The maximum consumption per head compatible with the given technological conditions, \(C\), is the level of \(c\) corresponding to \(g = 0\), that is, when the whole surplus is consumed:

\[
C = \frac{1 - a}{l}.
\]

Obviously, \(C\) equals the net output per unit of labor.

4. Wages and profits

The surplus is distributed to people as income. Assume that only two classes exist: workers and farmer-capitalists, receiving wages and profits, respectively. Let \(w\) denote the wage per hour in terms of corn, or, for short, the wage rate, and assume that wages are paid \textit{post factum}, that is, at the end of the production period. Let \(r\) denote the rate of profit, or the proportion of profits to the capital advanced at the beginning of the production period, which in the present case consists of seed-corn only. With free competition among laborers there will be a tendency for wage rates to equalize throughout the economy. Moreover, since all farmer-capitalists are assumed to have access to the same technology, this implies that there will also be a tendency for the rates of profit to become uniform throughout the economy, that is, the establishment of a \textit{general rate of profit}. On these premises \(w\) and \(r\) must satisfy the following equation (constant returns to scale need not be assumed):

\[
(1 + r)a + wl = 1, \tag{2.2}
\]

that is, one unit of gross output minus the capital used up \((a)\) constitutes the net output \((1 - a)\), which is distributed as wages \((wl)\), in proportion to labor, and as profits \((ra)\), in proportion to the value of capital. Equation
The wage rate (the rate of profit) depends both on the technical conditions of corn production, that is, $a$ and $l$, and on the rate of profit (the wage rate).

The general concept underlying the present approach to the theory of income distribution is that of normal or long-period positions of the economy. These are conceived as centers around which the economy is assumed to gravitate, given the competitive tendency toward a uniform rate of profit and a uniform rate of remuneration for each particular kind of "primary" input in the production technique, such as homogeneous labor in the present case.

A comparison of equations (2.1) and (2.2) reveals immediately that the relationship between $c$ and $g$ on the one hand and the relationship between $w$ and $r$ on the other have exactly the same form. This fact is referred to in the literature as the duality of the two relationships. In particular, the maximum rate of profit, $R$, corresponding to $w = 0$, is equal to the maximum growth rate,

$$R = \frac{1 - a}{a},$$

and the maximum wage rate, $W$, corresponding to $r = 0$, is equal to the maximum level of consumption per head,

$$W = \frac{1 - a}{l}.$$

5. Saving and investment

In Section 3 the problem of consumption and growth was dealt with by treating consumption per head $c$ or the growth rate $g$ as given. Similarly, in Section 4 the problem of distributing the net product between wages and profits was dealt with by considering the wage rate $w$ or, alternatively, the rate of profit $r$ as exogenously determined. These two spheres, that is, what is known as the quantity system and the price and distribution system, respectively, are connected via the explicit introduction of some hypothesis concerning saving and investment behavior. This problem will be dealt with in some detail in Chapter 15. Here it suffices to illustrate the connection under discussion in terms of a simple example. The example chosen is rather prominent in the literature on growth and
distribution, since according to the way in which it is interpreted, it is generally considered as conveying the essence of either the classical or the post-Keynesian economists’ view on the matter. Assume for simplicity that workers do not save, whereas capitalists save a proportion \( s \) of profits, where \( 0 < s \leq 1 \). The saving function can therefore be written

\[
S = sP,
\]

where \( S \) is total savings and \( P \) total profits. On the further assumption that total savings and total net investment are equal, we have

\[
I = sP,
\]

where \( I \) denotes investment, that is, in the present case, the amount of corn which will be added to the seed-corn advances at the beginning of the next production cycle to increase the overall level of production.

Dividing both sides of this equation by the capital (seed-corn) \( (K) \) advanced at the beginning of the current period gives

\[
g = sr \tag{2.3}
\]

or

\[
r = \frac{g}{s} \tag{2.3'}
\]

where \( g \) is the rate of accumulation \( (I/K) \), which is equal to the rate of growth of output, under constant returns to scale, and \( r \) is the rate of profit \( (P/K) \). Accordingly, the rate of growth is proportional to the rate of profit, with \( s \) as the proportionality factor.

The condition that saving equals investment can have two interpretations. According to the classical approach to the theory of accumulation from Adam Smith to David Ricardo this condition is equivalent to what became known as “Say’s Law”, that is, the proposition that there cannot be a “general glut” of commodities. Since all savings are assumed to become investment, the negative impact of saving on aggregate demand is instan-
taneously and exactly offset by the positive impact of additional investment. The classical view can therefore be characterized as follows: given the rate of profit, which in turn is seen to depend on the historically and socially given wage rate, and the capitalists’ propensity to save (and invest), the rate of accumulation is determined. A completely different point of view is taken by the post-Keynesian approach to the theory of growth and distribution advocated, among others, by Nicholas Kaldor and Joan Robinson. According to this approach it is investment which generates an equivalent amount of savings via changes in the distribution of income between wages and profits. The post-Keynesian view thus considers the
rate of profit as being determined by a given rate of growth of investment demand and given saving behavior.

Figure 2.1 illustrates the two views. According to the first interpretation it is the wage rate which is given at a level \( w = w^* \), whereas according to the second interpretation it is the rate of accumulation which is given at the level \( g = g^* \). From the \( w-r \) relationship (\( c-g \) relationship) depicted in quadrant I (II) we get the associated level of the rate of profit \( r = r^* \) (consumption per head \( c = c^* \)), which, via the accumulation function (saving function) (2.3) depicted in quadrant IV, translates into a rate of growth \( g = g^* \) (rate of profit \( r = r^* \)). In quadrant II (I) the \( c-g \) relationship (\( w-r \) relationship) is plotted, which gives the level of \( c = c^* \) (\( w = w^* \)) relative to the growth rate (rate of profit). Other things being equal, a higher (lower) wage rate is associated with a lower (higher) rate of profit, a lower (higher) rate of accumulation and thus rate of growth, and a higher (lower) level of consumption per head. Similarly, a higher (lower) rate of accumulation is associated with a lower (higher) level of consumption per head, a higher (lower) rate of profit, and a lower (higher) wage rate.

6. Choice of technique

Up til now it has been assumed that only one technique to produce corn, denoted as \((a, l)\), is known. We shall now relax this premise and assume that \( z \) different techniques, denoted as \((a_1, l_1), (a_2, l_2), \ldots, (a_z, l_z)\), are available from which the farmer-capitalists can choose. This is the problem of the choice of technique.

Let \( w^* \) and \( r^* \) be the ruling wage rate and rate of profit, respectively. If

\[
(1 + r^*)a_j + w^*l_j < 1 \quad (\geq 1)
\]
then technique \((a_j, l_j)\) is (is not) able to pay extra profits at \(w = w^*\) and \(r = r^*\). On the contrary, if
\[(1 + r^*)a_j + w^*l_j > 1 \quad (\leq 1)\]
then technique \((a_j, l_j)\) does (does not) incur extra costs at \(w = w^*\) and \(r = r^*\). If a technique is able to pay extra profits, then profit maximizing capitalists would operate it to obtain the extra income.

A long-period position is defined as a position such that nobody can get extra profits and the productive process can be performed; that is,
\[(1 + r^*)a_i + w^*l_i = 1 \quad \text{some } i \quad (2.4a)\]
\[(1 + r^*)a_j + w^*l_j \geq 1 \quad \text{each } j. \quad (2.4b)\]

Any technique which does not incur extra costs can be operated and at least one of them is; all the others cannot.

Obviously, the long period wage rate and rate of profit cannot both be given from outside. Let us assume therefore that the long period rate of profit \(r^*\) is given and let us determine the long period wage rate \(w^*\). In order to do so let us define \(w_j\) as the wage rate which can be paid with technique \((a_j, l_j)\) at the rate of profit \(r^*\). Hence
\[(1 + r^*)a_j + w_jl_j = 1 \quad \text{each } j. \quad (2.5)\]

Then, we obtain from (2.4) and (2.5)
\[(1 + r^*)a_i + w^*l_i = (1 + r^*)a_i + w_il_i \quad \text{some } i\]
\[(1 + r^*)a_j + w^*l_j \geq (1 + r^*)a_j + w_jl_j \quad \text{each } j\]

that is,
\[w^* = w_i \quad \text{some } i\]
\[w^* \geq w_j \quad \text{each } j.\]

Therefore
\[w^* = \max \{w_j | j \in J\},\]
where \(J\) is the set of all existing techniques. Another way to look at the same problem is that
\[w^* = \min \{w \in \mathbb{R} | w \geq w_j, j \in J\}.\]

If, on the contrary, the long period wage rate \(w^*\) is given, the long period rate of profit \(r^*\) is determined in a similar way. In fact we define \(r_j\) as the rate of profit which can be paid with technique \((a_j, l_j)\) at the wage
rate \( w^* \) and we follow the same procedure as above in order to obtain
\[
\begin{align*}
\omega &= r_i \quad \text{some } i \\
\omega &\geq r_j \quad \text{each } j.
\end{align*}
\]
Therefore
\[
\omega = \max \{ r_j | j \in J \},
\]
or
\[
\omega = \min \{ r \in \mathbb{R} | r \geq r_j, \quad j \in J \}.
\]
Thus, for a given wage rate (or, alternatively, a given rate of profit) the technique chosen in the long period coincides with the technique which yields the highest rate of profit (wage rate). Hence, if the (linear) \( w-r \) relationships relative to all techniques available are drawn in the same diagram, the \( w-r \) relationship relative to the whole economy is given by the outer envelope of all of them. The latter is known as the wage-profit frontier, or, for short, the wage frontier.
The wage-profit frontier \( w = F(r) \) can be obtained either as a solution of a minimum problem,
\[
\begin{align*}
F(r) &= \min w \\
\text{s. to } w &\geq \frac{1 - (1 + r)a_j}{l_j} \quad \text{all } j,
\end{align*}
\]
or as a maximum problem,
\[
\begin{align*}
F(r) &= \max w \\
\text{s. to } w &\in W,
\end{align*}
\]
where
\[
W = \left\{ w_j \in \mathbb{R} | w_j = \frac{1 - (1 + r)a_j}{l_j}, \quad j \in J \right\}.
\]
In the following chapters the first way to obtain the wage frontier will be called the direct approach and the second will be called the indirect approach. In Figure 2.2 it is assumed that there are four techniques: \( z = 4 \). Technique 3 is clearly inferior and will not be adopted irrespective of the level of the wage rate (rate of profit). The heavy line gives the relationship between the long-period levels of the wage rate in terms of corn and the rate of profit for the entire economy; it is constituted by the relevant segments of the \( w-r \) relationships associated with techniques 1, 2, and 4, respectively. For example, with the rate of profit given at a level \( r = r' \),
technique 2 will be chosen and the wage rate corresponding to this level of the rate of profit is \( w = w' \). It is also possible that at a given wage rate (rate of profit) more than one technique is cost minimizing. This is, for example, the case at a level of the wage rate \( w = w' \); at this level both techniques 1 and 2 are equiprofitable. The intersection of the \( w-r \) relationships associated with two techniques on the wage frontier is called a switchpoint.

7. Distribution and growth

In Section 5 it was argued that in steady-state analysis with planned investment equal to planned saving, the price and the quantity system are connected by the saving function. In this section we shall illustrate the involved relationship between income distribution and growth in the case in which there is a choice of technique in terms of a diagram which has the corn capital input per unit of labor employed, or capital intensity, \( k \), on the abscissa and the net output per unit of labor employed, or (net) labor productivity, \( y \), on the ordinate.

For technique \( i \) the two magnitudes are given by \( y_i = (1 - a_i)/l_i \) and \( k_i = a_i/l_i \), respectively, \( i = 1, 2, \ldots, z \). In Figure 2.3, in correspondence with Figure 2.2, four techniques are depicted, that is, \( z = 4 \). In this type of diagram the pair \((k, y)\) denotes technique \( i \). The problem of the choice of technique can now be illustrated as follows. Given the real wage rate, \( w \), from outside the system means fixing a value for \( w \) on the ordinate. Assume \( w = w' \). Then we have, from the equation giving the rate of profit
compatible with a given technique $i$ at wage rate $w'$,

$$r_i = \frac{y_i - w'}{k_i} \quad \text{all } i.$$

Thus, in Figure 2.3 the rate of profit associated with technique 2 is given by $\tan \alpha$, whereas the rate of profit associated with technique 3 is given by $\tan \beta$; the rates of profit associated with techniques 1 and 4 can easily be ascertained by the reader. Obviously, at the given wage rate $w = w'$, technique 2 yields the highest rate of profit, $r = r'$, and is thus the technique that minimizes costs: it defines the long-period position at the going wage rate $w'$. It cannot of course be ruled out that at a given level of the corn wage more than one technique satisfies the criterion of cost minimization. For example, in Figure 2.3 this is the case with a wage rate $w = w''$: both techniques 1 and 2 yield the same rate of profit, $r = r'' = \tan \gamma$, which is higher than the rates of profit attainable with any single other technique. Using the terminology introduced in Section 6, $w = w''$ represents a switchpoint wage rate at which the two techniques 1 and 2 are equiprofitable.²

With corn as the only capital good and the corn wage rate changing

² As the analysis in the preceding sections has made clear, the argument could also be carried out on the assumption that the rate of profit rather than the real wage rate is given from outside the system. Fixing the rate of profit means fixing the slope of the line intersecting the y-axis. Cost minimization then involves maximizing the real wage rate compatible with the technical alternatives, given $r$. 
hypothesized from zero to its maximum value compatible with the given technical alternatives, the set of cost-minimizing techniques can be expressed in terms of a *per capita* corn "production function." It is given by the bold lines, made up of techniques 1, 2, and 4. (Technique 3 is inferior irrespective of the level of the real wage rate or, alternatively, of the level of the rate of profit.)

The graph of Figure 2.3 can now be used to illustrate anew the difference between the "classical" and the "post-Keynesian" approach to the theory of distribution and growth; this is done in Figure 2.4. To simplify matters, assume that all wages are consumed and capitalists save a portion \( s \) of their profits as in Section 5. In classical analysis the real wage rate is given from outside the system of production, say at \( w = w' \). Cost minimization involves maximizing the rate of profit, that is, the slope of the straight line intersecting the \( y \)-axis at \( y = w' \) and passing through a point defining a technique. In this way also the growth rate is maximized, since it is determined by equation (2.3); in Figure 2.4 \( r = r' = \tan \alpha \). In contradistinction, in post-Keynesian analysis the rate of accumulation and growth is given exogenously, say at \( g = g' \), whereas the rate of profit and the wage rate are determined endogenously: \( r = r' = g'/s \) (cf. equation (2.3')), and \( w = w' \) is determined as in Figure 2.4 by the intersection with the \( y \)-axis of the highest straight line with slope \( r' = \tan \alpha \) passing through a point defining a technique.\(^3\)

\(^3\) For a more general and detailed treatment of the difference between the two theories see Chapter 15.
8. Historical notes

8.1. One-commodity models of the kind presented in this chapter, or of a similar kind, abound in the economics literature. They are a familiar work horse in the macroeconomic theory of growth and distribution. However, in much of the literature it is assumed that the capital used in production is not circulating but fixed capital of an everlasting, that is, perennial, nature. (See Chapter 7 and especially Chapter 9.)

8.2. Newman (1962), in a review article on the first part of the book by Sraffa (1960), has utilized the idea of changing the physical unit in terms of which a commodity is measured such that the gross output of the commodity equals unity, while the scale of its production remains unaltered. This device was designed to shed light on Sraffa’s claim that his argument does not rest “on a tacit assumption of constant returns in all industries” (Sraffa, 1960, p. v). (See also Subsection 8.3 of Chapter 13.)

8.3. In the above it has been assumed that wages are paid at the end of the production period, that is, post factum or post numerandum. The classical economists and Marx, however, reckoned wages as a part of total capital to be advanced at the beginning of the revolving (yearly) cycle of production. Accordingly, wages were taken to be paid ante factum or ante numerandum. For a justification of considering wages to be paid at the end of the period, see Sraffa (1960, pp. 9–10). (A further elaboration of Sraffa’s treatment of wages will be provided in the historical notes to Chapter 3.)

Clearly, if wages were to be paid ante factum, equation (2.2) in Section 4 would have to be replaced by

\[(1 + r)(a + wl) = 1,\]  \hspace{1cm} (2.2')

and the equation giving the \(w-r \) relationship would read:

\[w = \frac{1 - (1 + r)a}{1 + rl}.\]

Therefore, while with post factum payment of wages the \(w-r \) relationship relative to a given technique is a straight line (see Figure 2.1), with ante factum payment it is an hyperbola which is convex to the origin with asymptotes at \( r = -1 \) and \( w = -a/l \). A similar argument could be developed with regard to the \( c-g \) relationship dealt with in Section 3.

8.4. The constraint binding changes in the distributive variables, in particular the real wage rate and the rate of profit, was discovered (though
A one-commodity model

not consistently demonstrated) by Ricardo: "The greater the portion of the result of labour that is given to the labourer, the smaller must be the rate of profits, and vice versa" (Ricardo, Works, vol. VIII, p. 194). Similarly, Ricardo in his Notes on Malthus emphasized: "Mr. Malthus I believe would find it difficult to show that there can be any fall in the rate of profits unless there be a real rise in the value of labour when a larger proportion of the whole produce ... is devoted to the payment of wages" (Works, vol. II, pp. 61–2). Ricardo was thus able to dispel the idea, generated by Adam Smith's notion of price as a sum of wages and profits (and rents) (cf. WN, I, vi), that the wage rate and the rate of profit are determined independently of each other.

Since then the inverse relationship between the two distributive variables played an important role in long-period analysis of both classical and neoclassical descent. In more recent times it was referred to by Samuelson (1957), who later dubbed it "factor price frontier" (cf. Samuelson, 1962). Hicks (1965, p. 140, n. 1) objected that this term is unfortunate, since it is the earnings (quasi-rents) of the proprietors of capital goods rather than the rate of profit which are to be considered the "factor price" of capital (services). A comprehensive treatment of the problem under consideration within a classical framework of analysis was provided by Sraffa (1960); see also Pasinetti (1977), Lippi (1979), Mainwaring (1984), Goodwin and Punzo (1987), Woods (1990), Bidard (1991), and particularly Chapters 3 and 4 in this book. The relationship is also known as the "optimal transformation frontier" (Bruno, 1969) and the "efficiency frontier" (Hicks, 1973).

8.5. The duality of the w–r relationship and the c–q relationship in steady-state capital and growth theory has been demonstrated, among others, by Bruno (1969), Spaventa (1970), von Weizsäcker (1971), Hicks (1973), and in more general terms by Burmeister and Kuga (1970), Morishima (1971), Fujimoto (1975), and Craven (1979).

8.6. Historical notes on the post-Keynesian approach to the theory of distribution and growth will be provided in Chapter 15.

8.7. With a continuum of techniques available to produce corn, the production function need not, as in Figures 2.3 and 2.4, consist of a series of straight lines, but may be continuously differentiable (see exercises 9.4 and 9.5). The assumption that there is a continuously differentiable production function is prominent in the neoclassical literature on growth and distribution. See, for example, Solow's growth model (Solow, 1956), which centers around a function that is linear homogenous and can be written in terms of per capita output as a function of per capita capital.
9. Exercises

9.1. Determine the $w-r$ relationships of the Table 2.2 techniques.

9.2. Determine the wage frontier relative to the set of techniques described in exercise 9.1.

9.3. Show that the choice of technique as analyzed in Section 6 does not depend on the assumption of a finite number of techniques. **[Hint: Introduce a set of indices $J$; $J$ may be countable or uncountable; then techniques can be denoted as $(a_j, l)$ for $j \in J$. Then...]**

9.4. (Calculus) Let $J = \{ j \in \mathbb{R} | 0 \leq j \leq 5 \}$ and let $(a_j, l_j), j \in J$, be techniques such that $a_j = j/10$ and $l_j = 1/10j$. Determine the wage frontier. **[Hint: Use calculus and show first that if $0 \leq r \leq 0.25$, then $j = 10/(2(1 + r))$, whereas if $0.25 \leq r \leq 1.50$, then $j = 4$. Then...]**

9.5. (Calculus) Let $J = \{ j \in \mathbb{R} | j > 0 \}$ and let $(a_j, l_j), j \in J$, be techniques such that $a_j = j/10$ and $l_j = 1/10j$. Determine the wage frontier. **[Hint: See the hint for the previous exercise.]**

9.6. (Calculus) Calculate the $y-k$ relationship analogous to that of Figure 2.3, that is, the per capita production function, relative to the previous two exercises.

9.7 (Calculus) Calculate the $r-k$ relationship relative to exercises 9.4 and 9.5.

9.8. Show that if $(a, l)$ is a technique producing a surplus $(a < 1)$, then the amount of corn which needs to be sowned in order to obtain a surplus of exactly one unit of corn is $a/(1 - a)$. How much labor is employed?

9.9. As is well known, Adam Smith in the *The Wealth of Nations* distinguished between "productive" and "unproductive" labor. According to him, "some both of the gravest and most important, and some of the most frivolous professions" must be ranked among the latter: "churche..."
lawyers, physicians, men of letters of all kinds [including the authors of books on economics]; players, buffoons, musicians, opera-singers, opera-dancers, &c.” (WN, II.iii.2). In the present exercise call the labor provided by workers directly engaged in the production of corn “productive” and the labor provided by all other workers “unproductive”. Assume that each worker (productive or unproductive) consumes an amount $b$ of corn and that $(a + bi) < 1$. Determine the maximum number of unproductive workers for each productive worker employed.