Utility functions

In order to analyse statistical data on consumption we need to start from some hypothesis. The one that is both natural and fruitful is to assume the existence, for each (or for the average) consumer, of a continuous utility function

\[ u = f(x_1, x_2, \ldots, x_n) \]

and to give it certain properties that make it easy to maximize with the usual mathematical tools.

It will be assumed that there are only a finite number of commodities, say \( n \); and that each commodity can be characterized numerically. It will further be assumed that there exists a unit of measure for each commodity, for example in kilogrammes or in litres or simply in quantity. This being so, any assortment or bundle or packet of the \( n \) commodities can be numerically expressed as a vector with \( n \) components: \( (x_1, x_2, \ldots, x_n) \). Each component always refers to the same commodity.

It is useful to consider each commodity as being linked with an axis in the space of dimension, \( n \), of real numbers, \( R^n \). In this way all the properties of this space will be at our service. However, the commodity set will not be considered as covering the entire space, \( R^n \). We shall limit it by imposing 3 properties which define the commodity set:

1. A commodity may not be characterized by strictly negative numbers. Hence the commodity vector, \( x = (x_1, \ldots, x_n) \) can have no negative components.
2. Let \( x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \) be a bundle (or \( n \)-tuple) available to the consumer. Then any bundle of the form \( \alpha x^0 = (\alpha x^0_1, \ldots, \alpha x^0_n) \), \( 0 \leq \alpha \leq 1 \), can be extracted from this bundle. This is known as the property of divisibility.
Utility functions

(3) The commodity set contains the bundle, \((0, 0, \ldots, 0)\); moreover, if a bundle \(x^i\) belongs to the set then any bundle, \(x^k\), where \(x^k_i \geq x^i_i\), \(i = 1, \ldots, n\), belongs to the set. The commodity set is, therefore, unbounded from above.

The symbol \(\mathcal{U}\) can be interpreted as measuring the 'satisfaction' derived from the consumption of alternative consumption bundles. More precisely, \(\mathcal{U}\) is a number that facilitates the ordering of the commodity bundles according to the preferences of the consumer. The utility function is indeed a numerical representation of a preference ordering, in the sense that \(\mathcal{U}\) is a number associated with each possible consumption bundle, such that if one bundle, say \(x^0 = (x^0_1, x^0_2, \ldots, x^0_n)\), is preferred to another bundle, say \(x' = (x'_1, x'_2, \ldots, x'_n)\), the number associated with \(x^0\) is greater than the number associated with \(x'\).

1.1. Basic axioms on the preference relation

The first question to ask is: what are the conditions that make such a representation possible? In other words, we want to know what properties the consumer's preferences should have to make it possible to associate a number with each commodity bundle in the way described above.

It is intuitively clear that restrictive assumptions will have to be made: some possible irregular properties will have to be excluded from the analysis. The reader is requested to ask himself whether he is ready to consider the assumptions introduced below as acceptable, i.e. whether the inevitable loss in realism is not exaggerated.

Economic theory formulates these assumptions as axioms, i.e. as statements that are accepted as being true without proof. This means that theorists consider these as intuitively plausible foundations on which a scientific theory can be built. All sciences start from a set of axioms, as one has to start somewhere. There is no reason why economic science should be an exception to this rule.

Some definitions and a minimum of notation are required. The basic concepts are those of preference and indifference, which are supposed to have an immediate intuitive meaning. Let \(x^0\) and \(x'\) be two vectors representing two commodity bundles. We say that

\[ x^0 \preceq x' \] \(\iff\) \(x^0\) is not preferred to \(x'\)
Basic axioms on the preference relation

The relation ‘is not preferred to’ is taken as the elementary or primitive concept. The following relations are derived from it:

\[ x^0 \not\leq x' \text{ and } x' \not\leq x^0 \] means \( x^0 \sim x' \), i.e. \( x^0 \) is indifferent to \( x' \).

\[ x^0 \not\leq x' \text{ and not } x' \not\leq x^0 \] is written \( x' > x^0 \) and means \( x' \) is preferred to \( x^0 \).

With this notation, we are able to state the three basic axioms that are needed to establish the ‘existence’ of a continuous utility function. (The word existence is put between quotation marks to indicate that it is used in a strict sense, without any metaphysical connotation. This point is taken up below at the end of Section 1.4.)

A.1. **Axiom of comparability:** For any pair of commodity bundles \( x^0 \) and \( x' \) in the commodity set \( X \), the consumer is able to say \( x^0 \not\leq x' \) or \( x' \not\leq x^0 \).

Notice that the term ‘or’ means that at least one of the statements must hold. Both statements may therefore hold simultaneously, in which case we have \( x^0 \sim x' \).

A.1 aims at excluding the case where the consumer would be unable to make comparisons about some bundles. In other words, A.1 makes the preference relation complete.

A.2. **Axiom of transitivity:** \( x^0 \not\leq x' \text{ and } x' \not\leq x'' \) imply \( x^0 \not\leq x'' \).

If the bundle \( x^0 \) is not preferred to the bundle \( x' \), and the latter is not preferred to \( x'' \), then \( x^0 \) is not preferred to \( x'' \). In other words, the consumer’s preferences are consistent: he never contradicts himself.

Technically speaking, these two axioms give us a preordering (or ‘weak’ ordering). A preordering is indeed a binary relation which is reflexive and transitive. The binary relation \( \preceq \) just defined as transitive – is also reflexive, as clearly \( x \preceq x \) for any \( x \) in \( X \) according to A.1.

In order to follow standard terminology, we will use the word ‘ordering’ instead of ‘preordering’ in later sections, although this is technically incorrect.\(^1\)

A.2 excludes from the analysis a number of preference relations that may (perhaps frequently) exist and are not necessarily irrational.

\(^1\) A (strong) ‘ordering’ is such that \( x^0 R x' \) and \( x' R x^0 \) implies \( x^0 = x' \), where \( R \) is a binary relation on a set. In that case indifference is ruled out and replaced by equality.
EXERCISES

1.1. Discuss the example given by Pearce (1964, p. 20) of behaviour that seems to be inconsistent with A.2. A colleague is invited to dinner. At the end of the meal, he has to choose between a small and a large apple: he takes the small one (although he is still hungry). Then, choosing between a large pear and a small apple, he takes the pear (because he is so hungry). But when offered a large pear and a large apple, he takes the apple (because he prefers apples).

*Answer:* The preordering appears as intransitive: large apple $\succ$ large pear $\succ$ small apple $\succ$ large apple. Nevertheless, the choices are all perfectly rational: his first choice (the small apple rather than the large one) is based on the desire to be polite (i.e. not to be considered greedy); the two other choices are explained above. The example is perhaps not inconsistent with A.2 given the additional information that the colleague is hungry, and the fact that, in the first alternative, a sociological constraint (politeness) comes in. To take account of this constraint and make A.2 operational, one should recognize that the large apple in the first choice is not really on the same footing (the same commodity) as the large apple in the third choice.

1.2. Give a few examples of strong orderings.

A.1 and A.2 give a complete preordering. One is tempted to conclude that A.1 and A.2 are sufficient conditions for the existence of a utility function, i.e. that one can now associate a real number with each commodity bundle, in such a way that the natural order of the real numbers corresponds to the preordering. There is a complication, however: one can imagine preorderings that are complete but cannot be represented by a real-valued function.

Consider the following situation. I am a dipsomaniac. Consequently I systematically prefer any bundle that contains alcohol. Needless to say, if two bundles contain alcohol, I prefer the one which contains more of it. I am interested in the other commodities if and only if the two bundles contain the same quantity of alcohol.

Let there be two bundles $x^0$ and $x'$, each with two goods: beer ($x_1$) and bread ($x_2$).

If there is more beer in $x^0$ than in $x'$, i.e. if $x^0_1 > x'_1$, then I prefer $x^0$ to $x'$. Let there be a third bundle $x''$ such that $x''_1 = x^0_1$. I prefer $x''$ to $x^0$ only if $x''_2 > x^0_2$, i.e. if there is more bread in $x''$ than in $x^0$. 
Basic axioms on the preference relation

Figure 1.1. represents the situation.

\[ \begin{align*}
x^0 \text{ is preferred to } x' \text{ (which contains less beer) and } x'' \text{ is preferred to } x^0 \text{ (same quantity of beer but more bread). Notice that } x^0 \text{ is preferred to any bundle located in the shaded area and on the segment from } x^0 \text{ to the horizontal axis. Notice also that the dotted line represents bundles that are preferred to } x^0. \text{ As there are no points indifferent to } x^0, \text{ it is impossible to trace a continuous curve separating } x^0 \text{ from the bundles that are preferred to } x^0 \text{ and the bundles to which } x^0 \text{ is preferred. } x' \text{ and } x'' \text{ may come as close to } x^0 \text{ as one wishes without becoming indifferent to it.}
\end{align*} \]

Exercise

1.3. Does the example given above belong to the family of preorderings?

Hint: check for reflexivity and transitivity. You will find that the answer is: yes.

We have here an example of what has been called a lexicographic ordering (see Debreu, 1954). The adjective is well chosen, as commodity bundles are ordered in the same way as words are in a dictionary. To exclude it – and possible similar preorderings – it suffices to accept the following axiom:

A.3. Axiom of continuity: The set of bundles not preferred to \( x^0 \) and the set of bundles to which \( x^0 \) is not preferred are both closed in \( X \), for any \( x^0 \).
Take any point $x'$ not preferred to $x^0$. A.3 says that it is possible to let $x'$ come close enough to $x^0$ for $x^0$ not to be preferred to $x'$, i.e. to be indifferent to $x'$. This is exactly what was impossible in Figure 1.1. Indeed, the set of points not preferred to $x^0$ was not closed, i.e. it did not contain its boundaries. The same was true for the set to which $x^0$ was not preferred.

Debreu (1959, p. 60–63) shows that A.1, A.2 and A.3 are sufficient conditions for the existence of a real-valued utility function, which is a continuous function of the quantities consumed, such that $u(x^0) \leq u(x')$ when $x^0 \preceq x'$. Note that the Debreu theorem does not imply uniqueness of the utility function: any monotonic transformation of a utility function is also a utility function (i.e. we obtain ordinal rather than cardinal utility). This will be discussed further in Section 1.3.

1.2. Additional axioms

The result obtained in the preceding section is quite nice. However, the utility function whose existence is now beyond doubt still does not have all the desired properties. As was said in the first sentence of this chapter, we want to have a utility function because we want to maximize it, in the belief that the properties of this maximum are a good description of the observed behaviour of the consumer in the market. The commodity bundles purchased in the market will indeed be considered as "optimal" i.e. as bundles corresponding to a maximum of the utility function (given a budget constraint to be introduced later on).

To make this maximization feasible, we will introduce three additional axioms.

A.4. Axiom of dominance (or monotonicity): When two bundles $x^0$ and $x'$ in $X$ are such that $x^0$ dominates $x'$, then $x^0$ is preferred to $x'$.

Let $x^0 = (x^0_1, x^0_2)$ and $x' = (x'_1, x'_2)$. We say that $x^0$ dominates $x'$ if $x^0_1 \geq x'_1$ and $x^0_2 > x'_2$ or if $x^0_1 > x'_1$ and $x^0_2 \geq x'_2$.

In one word, the consumer is supposed to prefer the bundle that contains more of one of the two goods and not less of the other good. He always prefers more to less, hence the term 'monotonicity': this axiom makes $u$ a strictly increasing function of the quantities consumed. You will have
Additional axioms

noticed that the way lexicographic preferences were defined was consistent with the dominance axiom. As you did not protest at the time we discussed Figure 1.1, A.4 probably seems reasonable to you.

EXERCISES

1.4. Can you explain why A.4 implies the absence of saturation?

1.5. Draw the commodity set X, consisting of all bundles composed of x_1 and x_2. Locate a point x^0 = (x_i^0, x_j^0) and shade the area of points to which x^0 is preferred and the area of points preferred to x^0 according to A.4. If an indifference curve (the locus of all points indifferent to x^0) were to be drawn, what would be its shape, given that it has to pass through x^0?

Your knowledge of elementary textbook economics may have led you, while working out exercise 1.5, to draw an indifference curve through x^0 that is not only downward sloping but also convex to the origin. To be allowed to do this, however, we need a further axiom.

A.5. AXIOM OF STRICT CONVEXITY: If two bundles x' and x^0 are indifferent, then a linear combination of these bundles is preferred to x' and x^0.

Remember that a set S is said to be 'convex' if a line segment between any two points P_1 and P_2 in that set belongs entirely to it. If all the points

\[ P = aP_1 + (1 - a)P_2 \quad \text{where} \quad 0 \leq a \leq 1 \]

belong to S, then S is convex. The linear combination (or weighted mean) P indeed defines all the points on a straight line drawn between P_1 and P_2. Applied to a (convex) consumption set X, this means that preferences are convex if

\[ ax' + (1 - a)x^0 \succsim x^0 \quad \text{when} \quad x' \succ x^0 \quad (0 \leq a \leq 1) \quad (1.1) \]

or

\[ ax' + (1 - a)x^0 \succeq x^0 \quad \text{when} \quad x' \succ x^0 \quad (0 < a \leq 1). \quad (1.2) \]

Then the set of all points (in X) at least as desired as x^0 is a convex set. The same is true for the set of points preferred to x^0. Figures 1.2a and 1.2b illustrate this.
A.5 is however an axiom of strict convexity, and implies that
\[ ax' + (1 - a)x^0 > x^0 \quad \text{when} \quad x' \sim x^0 \quad (0 < a < 1). \]

The indifference curves have to be smooth (see Figure 1.2c), which is of course a stronger condition and corresponds to the sort of curve we had in mind when we started this discussion.

What does A.5 imply in terms of the utility function? We know that
\[ u(x') > u(x^0) \quad \text{when} \quad x' > x^0. \]

A.5 therefore implies
\[ u(ax' + (1 - a)x^0) > u(x^0) \quad \text{when} \quad x' \sim x^0, \]
\[ i.e. \ u(x^0) = u(x') \quad \text{and} \quad 0 < a < 1. \]
\[ \quad \text{(1.3)} \]
A utility function with property (1.3) is said to be strictly quasi-concave.

A.6. AXIOM OF DIFFERENTIABILITY.

We will finally assume that the strictly increasing (A.4) and quasi-concave (A.5) utility function is also twice differentiable. Together with A.4 and A.5, this assumption has important consequences for applied work, some of which may already be indicated.

Take the first order partial derivative \( \frac{\partial u}{\partial x_i} \), which is called the 'marginal utility' of good \( i \). Because of A.4, marginal utilities have always to be positive! When drawing a curve representing marginal utility of good \( i \) as a function of its quantity, we should therefore be careful not to cross the horizontal axis!

On the other hand, Young's theorem implies that the second-order derivatives are such that
\[ \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}, \]
\[ \quad \text{(1.4)} \]
so that the matrix of second-order derivatives, called 'Hessian matrix' is symmetric (see (1.9) below).

1.3. Ordinalism versus cardinalism

Now that we have a (well-behaved) utility function at our disposal, to represent a given preference ordering, a natural question is: is this representation unique?

The answer is: no. A whole family of transformations can do equally well, i.e. represent equally well the same preference ordering. To see this, it may be useful to start from the interpretation of $u$ as a measure of the satisfaction derived from the consumption of a commodity bundle, and to ask what sort of measure it could be. A digression on the notion of 'measure' and its properties is then in order. This digression opens with the following exercise.

**Exercise**

1.6. A 'measure' is a number associated with an entity. Let there be the entities A, B, C, D, E, F, G and the measures associated with each of these as indicated in column (1) of the following table.

<table>
<thead>
<tr>
<th>Entity</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
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</thead>
<tbody>
<tr>
<td>A</td>
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<td>B</td>
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<td>D</td>
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<td>E</td>
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<tr>
<td>F</td>
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<tr>
<td>G</td>
<td>10</td>
<td></td>
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</tr>
</tbody>
</table>

(a) Multiply the numbers in column (1) by 2, and write down the result in column (2). What sort of measures allow transformations of this type?

(b) Add the number 5 to the numbers of column (1) and write down the result in column (3). Example of measure where this sort of addition is currently made?

(c) Multiply column (1) by 2, add the number 3 to each result, and fill in column (4). If you have travelled from Anglo-Saxon to European countries, you have had to make this sort of transformation (to get an idea of the temperature).
Utility functions

(d) Column (1) is related to columns (2), (3) and (4) respectively by linear increasing transformations of the type \( y = ax + b \), \((a > 0)\), where the \(x\)'s are the measures in column (1). Fill in column (5), using a transformation of \( x \) that is non-linear.

A measure is said to be ‘cardinal’ when it is subject to linear (increasing) transformations only. It is ‘ordinal’ when all monotonic (increasing) transformations (which of course include linear transformations) are allowed. Clearly, to know whether a measure is cardinal or ordinal, one has to be informed about the type of permissible transformations, it being understood that all usual measures are subject to transformations.

What then about \( u \), our ‘measure’ of satisfaction? The authors who introduced this concept in economics (Menger, Gossen, Walras) implicitly or explicitly considered it as a cardinal measure. The immediate implication is that \( u \) is determined up to a linear (increasing) transformation. In terms of exercise 1.6, this means the following. Interpret the entities A, B, \ldots, G as different commodity bundles, and the numbers in column (1) as values of \( u \) associated with these bundles. The cardinal nature of \( u \) then implies that the numbers in columns (2), (3) and (4) express the satisfaction derived from the consumption of A, or B, etc. as well as do the numbers in column (1). In fact, any linear increasing transformation of column (1) does equally well. But column (5) could not replace column (1).

Cardinalism was soon very strongly objected to (in particular by Irving Fisher and Vilfredo Pareto). The decisive attack came in the thirties with the work of Hicks (1936), who showed that a rigorous analysis can be built on the assumption of a utility function defined up to a monotonic increasing transformation, which is of course much less restrictive. Defining \( u \) as an ordinal measure has at least two advantages. First, as a matter of scientific strategy, a less restrictive hypothesis is always to be preferred, because of the greater generality of the results derived from it. Second, it is not realistic to suppose that satisfaction is measurable in a cardinal way.

The more recent development of the axiomatic approach, defining the axioms necessary for establishing the existence of a well-behaved utility function, has laid the foundations for the ordinal approach. Indeed, any monotonic increasing transformation of \( u \) can serve as a representation of the underlying preference ordering. It is the preference
ordering that counts. In what follows, the utility function will therefore *always* be defined as ordinal, i.e. as determined up to an increasing monotonic transformation. The implications of this will become clear as we proceed.

**Exercises**

1.7. Suppose a utility function in terms of $x_1$ and $x_2$ is specified as

(1) \[ u = x_1 x_2 \]

Let there also be the following utility functions:

(2) \[ u = x_1^2 x_2^2 \]
(3) \[ u = x_1^3 x_2^3 \]
(4) \[ u = 4x_1 x_2 \]
(5) \[ u = 4x_1 x_2 + 17 \]
(6) \[ u = \log(x_1 x_2) \]

(a) Which functions are transformations of (1)?
(b) Which transformations can replace (1) from a cardinal point of view? From an ordinal point of view?
(c) Fill in the following table, in which each column corresponds to one of the six utility functions specified above.

<table>
<thead>
<tr>
<th>$x_1$</th>
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<th>(1)</th>
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</table>

1.8. Let $u$ be a utility function satisfying the axioms enumerated in Sections 1.1 and 1.2
(a) Show that, if $v$ is a monotonic increasing transformation of $u$, $v$ satisfies the axiom of dominance.
(b) What about the other axioms?
*Answer:* (a) By assumption, $v = F(u)$ such that $F' = dF/du > 0$. ($F$ designates the transformation). The axiom of dominance implies $\partial u/\partial x_1 > 0$. To show that $v$ satisfies the axiom, one has to prove that $\partial v/\partial x_i > 0$. Now $\partial v/\partial x_i = F' \cdot \partial u/\partial x_i$, i.e. the product of two
positive terms, which is positive. Q.E.D. Notice that $F' > 0$ implies that the transformation has to be strictly increasing. In other words, $F' \geq 0$ would not do.

The reader should be convinced by now that our axioms establish the existence of a class of utility functions, in the sense that, given a member of that class, any (strictly) increasing monotonic transformation of it also represents the underlying preference ordering. It is important then to verify to what extent the properties of a given utility function are invariant under monotonic increasing transformations, i.e. to what extent the properties of one utility function are shared by the other members of its class.

Let us consider the first and second order partial derivatives of $u$ and of $v = F(u)$, with $F' > 0$, where $v = F(u)$ designates the whole class of utility functions representing the same preference ordering, and set up the following table:

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
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</thead>
<tbody>
<tr>
<td>$\frac{\partial u}{\partial x_i}$</td>
<td>$\frac{\partial v}{\partial x_i} = F' \frac{\partial u}{\partial x_i}$</td>
</tr>
<tr>
<td>$\frac{\partial^2 u}{\partial x_i^2}$</td>
<td>$\frac{\partial^2 v}{\partial x_i^2} = F' \frac{\partial^2 u}{\partial x_i^2} + F'' \left( \frac{\partial u}{\partial x_i} \right)^2$</td>
</tr>
<tr>
<td>$\frac{\partial^2 u}{\partial x_i \partial x_j}$</td>
<td>$\frac{\partial^2 v}{\partial x_i \partial x_j} = F' \frac{\partial^2 u}{\partial x_i \partial x_j} + F'' \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$</td>
</tr>
</tbody>
</table>

where $F'' = d^2 F/du^2$. When $F$ is linear, $F'$ is a positive constant and $F'' = 0$.

Looking at the first order partial derivatives, we see that their absolute value is never invariant within the class of utility functions considered. The absolute value of the marginal utility of a good is not determined. But the sign of these derivatives is always invariant (and positive), whether $F$ is linear or not.

As for the second order direct derivatives, it is clear that their absolute value is not invariant, whether $F'' = 0$ or not. Their sign, however, is invariant if $F'' = 0$, i.e. if $u$ is cardinal. Indeed, $F'$ being positive, $\partial^2 u/\partial x_i^2$
will be negative if \( \partial^2 u / \partial x_i \partial x_j \) is negative, when \( F'' = 0 \). This is not necessarily so when \( F'' \neq 0 \). That is the reason why the famous 'law of decreasing marginal utilities' is cardinal.

**Exercises**

1.9. The preceding statements can be illustrated with the help of the following table, in which the utilities listed in column (1) are transformed linearly into those listed in column (2) and non-linearly into those of column (3). Compute the first and second differences corresponding to (2) and (3) respectively. Compare their absolute values and signs with those of the first and second differences of (1).

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<td>10</td>
<td>-1</td>
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<td>23</td>
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<td>-1</td>
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<td>225</td>
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</tr>
<tr>
<td>16</td>
<td>-1</td>
<td>35</td>
<td>33</td>
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<td>256</td>
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</tr>
<tr>
<td>16.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>36</td>
<td></td>
<td></td>
<td>272.25</td>
<td></td>
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</tr>
</tbody>
</table>

1.10. Use the transformations listed in exercise 1.7 to illustrate the preceding conclusions with respect to the invariance of the absolute value and the sign of the first and second order (direct and cross) partial derivatives. Which utility function obeys the law of decreasing marginal utilities?

Given that the absolute value of \( \partial u / \partial x_i \) is not determined, ordinal demand theorists are anxious to avoid the concept of 'marginal utility' and replace it by the concept of 'marginal rate of substitution' between goods \( i \) and \( j \), which is simply the ratio \( (\partial u / \partial x_i) / (\partial u / \partial x_j) \). The latter's absolute value (just as its sign) is indeed invariant under monotonous transformations of \( u \) (why?).

For analogous reasons, orthodox ordinal theorists avoid concepts defined in terms of the absolute value or the sign of the second order derivatives of \( u \). One example is the concept of independence or 'additivity', defined by the property that \( \partial^2 u / \partial x_i \partial x_j = 0 \). (Section 3.1 will be
devoted to it.) A good deal of recent applied econometric analysis in the field is however based on utility functions specified as being additive, for reasons to be explored. This indicates that econometricians find it worthwhile to investigate and to test empirically certain cardinal properties of a given utility function, although these properties are known not to be invariant under monotonic transformations. This is probably a transitory state of affairs: a day will come when these properties will have been redefined in ordinal terms. Some progress in that direction is reported in Section 3.4. Another example is the law of decreasing marginal utilities. Again, applied econometricians are happy to discover that their estimates of the second order partial direct derivatives of the utility function (on which their estimates are based) show up with a negative sign. They know that some transformation(s) may change this sign and are careful not to utilize these members of the class of utility functions considered.

1.4. Maximization of the utility function

The quantities purchased by a consumer are supposed to be optimal quantities, i.e. quantities determined by maximizing his utility function under a budget constraint. Formally, he is thus supposed to maximize

\[ u = f(x_1, \ldots, x_n) \]

subject to the linear constraint

\[ \sum_i p_i x_i = y \]  \hspace{1cm} (1.5)

where \( p_i \) represents the price of the \( i \)th commodity and \( y \) designates his total expenditures, called 'income'. All prices are supposed to be given: the consumer cannot influence them. \( y \) is fixed: the problem of how much to spend (and therefore how much to save) out of disposable income is not taken up here, to concentrate attention on the allocation of a given budget among \( n \) goods. In the final chapter of Part II, we will try to solve the difficult problem of simultaneously determining total expenditures \( (y) \) and their allocation among \( n \) goods.

Here, we reduce the analysis to the simple classical calculus problem of finding a constrained maximum of \( u \), the constraint being linear. The successive steps are: (a) find the first order or necessary conditions for a local interior maximum; (b) verify the second order conditions for a local maximum; (c) make sure that the conditions for a global maximum
Maximization of the utility function

are satisfied. Our interest is indeed in the global maximum which, after all, is the maximum.\textsuperscript{2}

To find the first order conditions for a local maximum, we form the Lagrangian function

\[ L = u - \lambda \left( \sum_i p_i x_i - y \right) \]  \hspace{1cm} (1.6)

where \( \lambda \) is a Lagrange multiplier and we differentiate \( L \) with respect to \( x_i \) and \( \lambda \), to obtain

\[ \frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i \]

\[ \frac{\partial L}{\partial \lambda} = \sum_i p_i x_i - y. \]

Putting all derivatives equal to zero, we obtain the \((n + 1)\) first order conditions:

\[ \frac{\partial u}{\partial x_i} = \lambda p_i \hspace{1cm} (i = 1, 2, \ldots, n) \]

\[ \sum_i p_i x_i = y. \]  \hspace{1cm} (1.7)

On the assumption that the conditions for a global maximum are satisfied – which is a valid assumption as we shall see in a moment – the solution of system (1.7) provides us with the \( n \) optimal values of \( x_i \) and the equilibrium value of \( \lambda \). The \( n \) equilibrium values of \( x_i \) appear as functions of all prices and of \( y \).\textsuperscript{3} These functions are the demand functions, which describe the behaviour of the consumer in the market. These are therefore the functions we are interested in, whose properties we will analyse in Chapter II and whose parameters we will want to estimate in subsequent chapters.

Notice that the first \( n \) equations can be written as

\[ \frac{\partial u/\partial x_1}{p_1} = \frac{\partial u/\partial x_2}{p_2} = \cdots = \frac{\partial u/\partial x_n}{p_n} = \lambda \]  \hspace{1cm} (1.8a)

\textsuperscript{2} For a general exposition of these conditions, see Intriligator (1971) or Lancaster (1968).

\textsuperscript{3} This can be proved, using the implicit function theorem. See e.g. Intriligator (1971, Chapters 3 and 7).
which expresses the well-known cardinal principle that, at equilibrium, all marginal utilities divided by the corresponding prices are equal. In an ordinal approach, these conditions are rewritten as

\[
\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{p_i}{p_j}
\]

(1.8b)

where the ratio at the left hand side is interpreted as a marginal rate of substitution (invariant under monotonic transformations of \(u\)).

**Exercises**

1.11. Maximize the utility functions (1), (2), (4), (5) and (6) of exercise 1.7, subject to the constraint \(p_1x_1 + p_2x_2 = y\), and derive in each case the 2 demand equations and the equilibrium value of \(\lambda\).

*Answer:* For \(u = x_1x_2\), the first order conditions are

\[
x_2 = \lambda p_1
\]

\[
x_1 = \lambda p_2
\]

\(p_1x_1 + p_2x_2 = y\).

To solve this system, insert the first two conditions in the budget constraint, to obtain \(p_1(\lambda p_2) + p_2(\lambda p_1) = y\). The equilibrium value of \(\lambda\) is therefore

\[
\lambda^o = \frac{y}{2p_1p_2}.
\]

Inserting this value in the first two conditions, we obtain the system of demand equations:

\[
x_1^o = \frac{y}{2p_1}
\]

\[
x_2^o = \frac{y}{2p_2}.
\]

The maximization of the other utility functions leads to exactly the same solutions. This is not at all surprising, as these utility functions all belong to the same class (i.e. are monotonic transformations of each other) and therefore represent the same preference ordering. One and the same preference ordering should lead to the same behaviour in the market, whatever its numerical representation. The general proof of this central result is given in exercise 1.15 below.
Maximization of the utility function

The demand functions just obtained have a very special and uncommon feature: the quantity demanded of each good depends only upon its own price. Normally all prices appear in each demand equation. We shall have to wait until Chapter III (Section 3.5) to find out why these demand equations have this special property.

1.12. Maximize \( u = x_1x_2 \) given the constraint \( 2x_1 + 5x_2 = 100 \) and draw (on graph paper) the map of indifference curves representing \( u \), the budget constraint and the equilibrium point.

Hint: Remember that \( x_1^0 = 100/2 \times (2) \) and \( x_2^0 = 100/2 \times (5) \), and that the indifference curve passing through the equilibrium point has the utility index \( 25 \times 10 = 250 \) associated with it.

1.13. (a) Show that the marginal rate of substitution is equal to minus the slope of the indifference curve to which it relates and, at equilibrium, to minus the slope of the budget line.

(b) What is the equation of the slope of the indifference curves defined by the utility functions (1), (2), (4) and (6) of exercise 1.7? Do you notice anything special?

Answer: (a) An indifference curve is defined by the property that \( du = 0 \). From

\[
\frac{du}{dx_1} x_1 + \frac{du}{dx_2} x_2 = 0,
\]

we derive

\[
-\frac{du}{dx_2} x_2 = \frac{du}{dx_1} x_1,
\]

\[
\frac{dx_2}{dx_1} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}.
\]

d\( x_2 \)/d\( x_1 \) is the slope of the indifference curve. At equilibrium (see the first order conditions above),

\[
\frac{\partial u}{\partial x_1} = \frac{p_1}{p_2},
\]

\[
\frac{\partial u}{\partial x_2} = \frac{p_1}{p_2},
\]

while the budget constraint can be written \( x_2 = (y - p_1x_1) / p_2 \). The slope of the budget line is therefore \(-p_1/p_2\).
Utility functions

(b) The slopes are all equal to \( x_2/x_1 \). Along a line passing through the origin, i.e. for a given ratio \( x_2/x_1 \), all indifference curves have the same slope. These utility functions are thus 'homothetic'. On this, see Section 3.5.

1.14. Take the utility function \( u = x_1x_2 \) and modify it in the following way. Replace \( x_i \) by \((x_i - \gamma_i)\) where \( \gamma_i \) is a constant, and allow each variable to have an exponent \( \beta_i \neq 1 \). You get \( u = (x_1 - \gamma_1)^{\beta_1} (x_2 - \gamma_2)^{\beta_2} \). Apply a logarithmic transformation to get another member of its class and generalize to \( n \) goods. You end up with \( u = \sum_i \beta_i \log (x_i - \gamma_i) \). This is the well-known Stone-Geary utility function (about which more will be said in Chapter IV). Derive the system of demand equations, on the normalizing assumption that \( \sum_i \beta_i = 1 \).

\[ \beta_i = \frac{\lambda p_i(x_i - \gamma_i)}{\sum_i \beta_i} = \frac{\lambda p_i(x_i - \gamma_i)}{\lambda \sum_i p_i(x_i - \gamma_i)} \]

\[ \lambda = \frac{1}{y - \sum_i p_i \gamma_i} \]

(No confusion should arise from the fact that, from now on, we write the equilibrium \( \lambda^0 \) as \( \lambda \) and \( x_i^0 \) as \( x_i \) to save on notation.) Substituting, we find

\[ \beta_i = \frac{p_i(x_i - \gamma_i)}{y - \sum_j p_j \gamma_j} \]

or

\[ x_i = \gamma_i + \frac{\beta_i}{p_i} (y - \sum_j p_j \gamma_j) \]

which is the famous 'linear expenditure system' to be analysed extensively in Chapter IV. Now, each quantity demanded is a function of \( y \) and of all prices (the price \( p_i \) and the prices of all other goods).

Before taking up the second order conditions for a (local) maximum, we want to pause a moment and ask what may be the meaning, from
Maximization of the utility function

the point of view of the economist, of the (somewhat mysterious) Lagrange multiplier $\lambda$. It is possible and useful to give it an economic interpretation.

You will have noticed that $\lambda$ appears as a factor of proportionality equal – at equilibrium – at the marginal utility of any good divided by its price. At equilibrium, $\lambda$ is therefore the utility provided by the 'last dollar' spent, or 'the marginal utility of money' (in the terminology introduced by Alfred Marshall). To see this, consider that

$$\frac{\partial u}{\partial x_i} = \lambda p_i$$

may be written as

$$\frac{\partial u}{\partial (p_i x_i)} = \lambda$$

as $p_i$ is a constant. The product $(p_i x_i)$ is the expenditure on good $i$. At equilibrium, an additional dollar spent on any good ($i$ may be any commodity) therefore provides the same increase in utility $\lambda$. We can therefore say that $\lambda$ is the change in the maximized value of utility as income changes, or

$$\lambda = \frac{\partial u}{\partial y}$$

at equilibrium.

It is also illuminating to interpret $\lambda$ as a number that converts money into utility, as

$$\lambda = \frac{1}{p_i} \frac{\partial u}{\partial x_i}$$

and $1/p_i$ is the number of units of good $i$ that can be bought with one dollar (one monetary unit).

Finally, we notice that the equilibrium $\lambda$ is a function of income and all prices (see exercises 1.11 and 1.14) and that this function is homogeneous of degree minus one in prices and income. (Check in exercises 1.11 and 1.14 by multiplying all prices and income by a constant). This property of $\lambda$ is related to the fact (to be established in the next chapter)

---

*The proof will be given in exercise 1.22 using the 'indirect' utility function.*
that the quantities demanded remain unchanged when all prices increase or decrease proportionally, i.e. that the demand functions are homogeneous of degree zero. The solution of the first order conditions being unchanged, \( \partial u/\partial x_i \) remains unaffected by a proportional change in prices and income, so that \( \lambda \) has to be divided by the constant by which \( p_i \) is multiplied for \( \partial u/\partial x_i = \lambda p_i \) to remain true.

**Exercises**

1.15. \( u \) may be replaced by a monotonic transformation of it. It should be true, therefore, that the demand equations obtained by maximizing a transformation of \( u \) are the same as those obtained by maximizing \( u \). In working out exercise 1.11, we saw that this was indeed the case. You are well equipped, by now, to give a general proof.

**Answer:** Constrained maximization of \( v = F(u) \), with \( F' > 0 \), leads to the first order conditions

\[
\frac{\partial v}{\partial x_i} - \lambda^* p_i = 0 \quad \text{and} \quad \sum p_i x_i = y,
\]

where \( \lambda^* \) is the Lagrange multiplier associated with the maximization of \( v \). The first \( n \) conditions can be rewritten as

\[
F' \frac{\partial u}{\partial x_i} - \lambda^* p_i = 0
\]

or

\[
\frac{\partial u}{\partial x_i} - \frac{\lambda^*}{F'} p_i = 0.
\]

As both \( \lambda \) (utilized in the maximization of \( u \)) and \( \lambda^* \) have to be positive (why?), we can replace \( \lambda^*/F' \) by \( \lambda \), since \( \lambda^* = F'(\partial u/\partial y) = F'\lambda \). We therefore end up with the same first order conditions as those obtained by maximizing \( u \). The solution of these conditions, i.e. the system of demand equations, must therefore be the same.

Q.E.D.

1.16. (a) Derive the equilibrium values of \( \lambda^* \) associated with each of the utility functions (2), (4), (5) and (6) of exercise 1.7, given that you know the \( \lambda \) associated with \( u = x_1 x_2 \) (exercise 1.11).
Maximization of the utility function

(b) Does the evolution of \( \lambda \) or \( \lambda^* \) as a function of \( y \), in exercise 1.11, appear realistic to you? Compare with exercise 1.14.
(c) Are the absolute value and the sign of the marginal utility of money invariant under monotonic increasing transformations of the utility function?
(d) What about the absolute value and the sign of the derivative of the marginal utility of income with respect to income?
(e) What about the elasticity of the marginal utility of income with respect to income \( \partial \lambda / \partial y \cdot y / \lambda \)?

The entire preceding discussion has been based on the assumption that the second order conditions for a local maximum and the conditions for a global maximum are satisfied. The time has come to look at these conditions, which will be given without proof.

Sufficient conditions for the second order conditions are as follows: for a constrained maximum, with one constraint, the determinant of the bordered Hessian should have the sign of \( (-1)^n \), where \( n \) is the number of variables, the largest principal minor should have a sign opposite to this, and successively smaller principal minors should alternate in sign, down to the principal minor of order 2.

We already know that the 'Hessian' is the symmetric matrix of second order partial derivatives of the utility function. The bordered Hessian is defined here as

\[
\begin{pmatrix}
0 & p_1 & \cdots & p_n \\
p_1 & u_{11} & \cdots & u_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
p_n & u_{n1} & \cdots & u_{nn}
\end{pmatrix}
\]

where

\[ u_{ii} = \frac{\partial^2 u}{\partial x_i^2}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \]

We see that the Hessian \( U \)

\[
U = \begin{bmatrix}
u_{11} & \cdots & u_{1n} \\
\vdots & \ddots & \vdots \\
u_{n1} & \cdots & u_{nn}
\end{bmatrix}
\]

(1.9)
Utility functions

is bordered by a row and a column containing a zero and the partial derivatives (with respect to $x_i$) of the budget constraint $\sum p_i x_i - y = 0$.

The principal minors referred to are obtained by deleting the last $1, 2, \ldots, r \ldots$ rows and columns of the bordered Hessian.

In the particular case where $n = 2$, the sufficient condition is thus that

$$\begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11} & u_{12} \\ p_2 & u_{21} & u_{22} \end{vmatrix} = 2u_{12}p_1p_2 - p_1^2u_{22} - p_2^2u_{11} > 0 \quad (1.10)$$

and

$$\begin{vmatrix} 0 & p_1 \\ p_1 & u_{11} \end{vmatrix} = -p_1^2 < 0.$$

Looking more closely at the determinant of the bordered Hessian for $n = 2$, we see that the sign condition can be rewritten as

$$p_1^2u_{22} + p_2^2u_{11} - 2u_{12}p_1p_2 < 0. \quad (1.11)$$

Now, a well known mathematical result is that, if the utility function is (strictly) concave, then the Hessian $U$ is negative semi-definite\(^5\) so that (1.11) is satisfied. None of our axioms, however, implies the concavity of the utility function. The axiom of convexity guarantees that the utility function is quasi-concave (or 'concave-contoured'). Remember that quasi-concavity is defined according to (1.3) as

$$u(ax' + (1-a)x^0) > au(x') = u(x')$$

which means that an indifference curve (i.e. a contour) is a lower boundary of a convex set\(^6\). (Strict) concavity, on the other hand, is defined as

$$u(ax' + (1-a)x^0) > au(x') + (1-a)u(x^0) \quad \text{for} \quad x' \neq x^0. \quad (1.12)$$

In (1.12), it is not supposed that $x'$ and $x^0$ are equally preferred bundles. Concavity therefore depends both on the contours and on the way utility changes from contour to contour: it should change in such a way

---

\(^5\) I.e. $z'Uz \leq 0$ for any vector $z \neq 0$. See e.g. H.G. Eggleston, *Convexity*, Cambridge University Press, 1958, p. 51.

\(^6\) Quasi-concavity implies $z'Uz \leq 0$, where $z$ is not an arbitrary vector but the difference between $x'$ and $x^0$, these being equally preferred bundles.
Maximization of the utility function

that marginal utilities are decreasing. In applied work we shall often introduce the assumption of concavity.

However, concavity is too restrictive for theoretical purposes. It is more general to assume that the indifference curves are convex from below (i.e. that the utility function is quasi-concave). The sufficient conditions are then also satisfied. To show this, remember that the slope of an indifference curve is equal to\(^7\)

\[
\frac{dx_2}{dx_1} = -\frac{u_4}{u_2}
\]

(1.13)

where \(u_i = \frac{\partial u}{\partial x_i}\). Convexity implies \(d^2x_2/dx_1^2 > 0\). Now (please check)

\[
\frac{d^2x_2}{dx_1^2} = -\frac{1}{u_2}(u_{14}u_2^2 - 2u_{12}u_1u_2 + u_{22}u_1^2)
\]

\[= -\frac{1}{u_2p_1^2}(u_{14}p_2^3 - 2u_{12}p_1p_2 + u_{22}p_1^2)
\]

(1.14)

on substituting \(u_i = p_1u_2/p_2\). \(d^2x_2/dx_1^2\) is positive if and only if (1.11) is satisfied, since \(u_2p_2^2\) is positive.

**Exercise**

1.17. Verify that condition (1.11) is satisfied in exercises 1.11 and 1.14, even when the second order direct derivatives of \(u\) are not negative.

It remains for us to consider the conditions for a global maximum. For a problem concerned with maximizing a continuous function \(f(x)\) over a closed feasible set \(K\), every local maximum is also a global maximum if: (a) \(f\) is a concave function and (b) \(K\) is a convex set. And if \(f\) is strictly concave over a convex feasible set the global maximum is unique. Furthermore, it is sufficient for satisfaction of the global optimum conditions for a maximum that \(f(x)\) be a monotonic increasing transformation of a concave function, and \(K\) a convex set (Lancaster, 1968, p. 17–19).

As any quasi-concave function \(f(x)\) which is an increasing function of every component of the vector \(x\) can be expressed as a monotonic increasing transformation of some concave function, all utility functions obeying our axioms are monotonic increasing transformations of some concave utility function. Any local maximum is therefore also a global maximum.

\(^7\) See exercise 1.13.
EXERCISE
1.18. Verify that the utility functions (1), (2), (4) and (5) of exercise 1.7 are (a) quasi-concave and (b) monotonic increasing transformations of the strictly concave utility function (6) $u = \log (x_1 x_2)$.

*Hint:* apply (1.12) and (1.3).

To conclude this section, it may be useful to say a word about the *epistemology* of utility maximization. Some readers may have the impression that the preceding exposition is purely 'formal', without any relation to 'reality'. They may wonder whether the utility function exists 'really' — (not just formally); if it does, whether the consumer wants to maximize it; and even if he does, whether he would be able to do so.

The answer to each of these questions is probably: no. Negative answers to these sorts of questions are not considered relevant by the economist. The utility function is a formal concept useful to the economist, not to the consumer. By postulating (and possibly specifying) a utility function, the economist wants to create a tool useful for a correct description of observed consumer behaviour in the market and a reasonably good forecast of future behaviour. In the present state of the art, the best descriptions and therefore the best forecasts are obtained when using the assumption (or the implications) of utility maximization (as we shall see in later chapters); this fact is sufficient to reject criticisms based on the alleged 'formal' or 'unrealistic' character of our approach. This fact also confirms a more general observation, according to which 'reality', as it appears in the available statistical data, becomes intelligible only to the extent that it is interpreted with the help of a formal hypothesis that is *imposed* on the observed data. 'Reality' has to be remodeled to become interesting. Needless to say, further progress of economic science may (and probably will) one day put up an assumption — and corresponding axioms — that leads to even better descriptions and forecasts. At the moment, many refinements are already available, as will become clear in Part II of this book where we shall dynamize the utility function and put the problem in an intertemporal framework.

In the limit, one may say that the utility function exists because we postulate it. Its maximization is the logical consequence of our axioms. It is the economist who maximizes utility to find the 'optimal' quantities corresponding to the quantities that the consumer effectively purchases.
Indirect utility functions

in the market. The optimization technique is thus simply a procedure that is utilized because it works, i.e. because it leads to operational hypotheses which turn out to be valid. Its justification lies in the conclusions that can be derived from it. (For an analogous argument, see Kuenne (1963, p. 16).)

These conclusions obviously refer to the demand equations, and take the form of restrictions to be imposed on these equations. Chapter II derives the 'general' restrictions that result from utility maximization as such. Chapter III discusses the additional 'particular' restrictions (as we shall call them) that result from the use of utility functions specified as having particular properties, and shows how to incorporate these restrictions in the empirical estimation of price and income elasticities. In this philosophy, applied consumption analysis appears then as the art of constructing and effectively utilizing interesting theoretical restrictions in the (econometric) estimation of demand equations. That this art is worth practising and developing is indicated by the observation that stronger (more particular that is) restrictions produce more precise estimations and better forecasts.

1.5. Indirect utility functions

All utility functions introduced above are 'direct': they have \( x_i \) (\( i = 1, \ldots, n \)) as arguments. We know that constrained maximization leads to a system of demand equations of the type

\[
x^*_i = \phi_i(p_1, \ldots, p_m, y).
\]

When we replace \( x_i \) by the optimal \( x^*_i \) in the direct utility function, we obtain an alternative description of a given preference ordering, called the indirect utility function, which we write as

\[
u^* = f[\phi_1(p_1, \ldots, p_m, y), \phi_2(p_1, \ldots, p_m, y), \ldots, \phi_n(p_1, \ldots, p_m, y)] \tag{1.15}
\]

for \( i = 1, \ldots, n \). The indirect utility function has prices and income as arguments.

**Exercises**

1.19. Write down the indirect utility functions corresponding to the direct utility functions (1), (2), (4), (5) and (6) of exercise 1.7.
Utility functions

1.20. What is the indirect utility function corresponding to the Stone-Geary function?

*Hint:* See exercise 1.14.

We now investigate the properties of $f^*(p_1, \ldots, p_n, y)$. First of all, we want to emphasize that $f^*$ is obtained by substituting $x_i^0$ for $x_i$ in $f(x_1, \ldots, x_n)$, $x_i^0$ being optimal quantities that maximize $u$, $u^*$ represents the *highest* utility that may be obtained with alternative (given) prices and incomes.

Another interesting property is the following. We will show that the demand functions $\phi_i$ are homogeneous of degree zero (in income and prices). Consequently, the indirect utility function is also *homogeneous of degree zero*: as a proportional change in all prices and income does not affect $x_i^0$, it cannot affect $u^*$ either. The price-income indifference surfaces

$$f^*(p_1, \ldots, p_n, y) = \text{constant}$$

can therefore be represented by cones whose summit lies on the origin, in a three-dimensional diagram with $y$ on the vertical axis and prices on the horizontal axes (see Figure 1.3).

![Fig. 1.3.](image)

Figure 1.3 represents three indifference surfaces corresponding to three different constant values of $u^*$.

**Exercise**

1.21. Verify that the indirect utility functions derived in exercises 1.19 and 1.20 are indeed homogeneous of degree zero in income and prices.
Furthermore, there is a duality relation\(^8\) between \(f(x_1, \ldots, x_n)\) and \(f^*(p_1, \ldots, p_n, y)\). (This is not surprising, given that both represent the same preference ordering.) Maximization of \(f\) with respect to the \(x\)'s, with given prices and income, leads to the same demand equations as minimization of \(f^*\) with respect to prices and income, with given quantities. This property is the more useful as we can, to derive the demand equations from \(f^*(p_1, \ldots, p_n, y)\), apply a formula known as Roy's identity\(^9\). Roy derives his identity in the following way. At equilibrium, we must have
\[
\text{du}^* = 0 \quad \text{and} \quad \sum_i x_i \text{d}p_i = \text{dy},
\]
or
\[
\frac{\partial f^*}{\partial p_1} \text{d}p_1 + \frac{\partial f^*}{\partial p_2} \text{d}p_2 + \ldots + \frac{\partial f^*}{\partial p_n} \text{d}p_n = -\frac{\partial f^*}{\partial y} \text{d}y
\]
and
\[
x_1^0 \text{d}p_1 + x_2^0 \text{d}p_2 + \ldots + x_n^0 \text{d}p_n = \text{dy}
\]
which implies
\[
\frac{\partial f^*/\partial p_i}{x_i^0} = \ldots = \frac{\partial f^*/\partial p_n}{x_n^0} = -\frac{\partial f^*}{\partial y}
\]
or
\[
x_i^0 = -\frac{\partial f^*/\partial p_i}{\partial f^*/\partial y}.
\]

Once \(f^*\) has been specified, it suffices to apply this identity to obtain the demand functions.

**Exercises**

1.22. In Section 1.4 we have interpreted \(\lambda\) as being the marginal utility of money or \(\partial u/\partial y\) at equilibrium. The time has come to show that indeed \(\lambda^0 = \partial u^*/\partial y\).

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\(^8\) A recent treatment of this and related duality relations is to be found in Bronsard (1971) and Lau (1972).

Answer: Differentiating $u^*$ with respect to $y$, we have
\[
\frac{\partial u^*}{\partial y} = \sum_i \frac{\partial f^*}{\partial x_i} \frac{\partial x_i}{\partial y}.
\]

Differentiating the budget constraint with respect to $y$, we obtain
\[
\sum_i p_i \frac{\partial x_i}{\partial y} = 1 \quad \text{or} \quad \lambda^0 \sum_i p_i \frac{\partial x_i}{\partial y} = \lambda^0.
\]

Subtracting the second result from the first, we find
\[
\frac{\partial u^*}{\partial y} = \sum_i \frac{\partial f^*}{\partial x_i} \frac{\partial x_i}{\partial y} - \lambda^0 \sum_i p_i \frac{\partial x_i}{\partial y} + \lambda^0 = \lambda^0 + \sum_i \left( \frac{\partial f^*}{\partial x_i} - \lambda^0 p_i \right) \frac{\partial x_i}{\partial y}.
\]

As $\frac{\partial f^*}{\partial x_i} - \lambda^0 p_i = 0$ for all $i$, $\frac{\partial u^*}{\partial y} = \lambda^0$. Q.E.D.

1.23. Apply Roy's identity to the indirect utility functions obtained in exercises 1.19 and 1.20 to verify that you obtain the same demand equations as in exercises 1.11 and 1.14.

1.24. Let $u^* = \sum_i a_i (y/p)_i^b$ be the indirect 'addilog' utility function as specified by Houthakker in (1960b).

Derive the demand equations. Are these linear? If not, could you transform the demand equations into linear expressions (suitable for econometric estimation)?

Answer: From Roy's identity, the demand equations are
\[
x_i = \frac{a_i b_i y^p p_i^{-b_i -1}}{\sum_j a_j b_j y^p p_j^{-b_j -1}}.
\]

These are non-linear expressions. It would be very hard – and probably impossible – to imagine a workable method of estimating the coefficients $a_i$ and $b_i$. The non-linearity is due to the presence of the sum in the denominator. How can we get rid of it?

Notice that the same sum appears in each demand equation. Why not divide one equation by another? One gets
\[
\frac{x_i}{x_j} = \frac{a_i b_i y^p p_i^{-b_i -1}}{a_j b_j y^p p_j^{-b_j -1}}.
\]
Indirect utility functions

This is still non-linear, but a few additional manipulations suffice to give us a linear expression. Multiplying the numerator and the denominator by \( y \) we obtain

\[
\frac{x^i}{x_j} = \frac{a_i b_i (y/p_j)^{b_i + 1}}{a_j b_j (y/p_j)^{b_j + 1}}
\]

and, taking logarithms,

\[
(\log x_i - \log x_j) = (\log a_i b_i - \log a_j b_j) + (1 + b_i) \log (y/p_i)
\]

\[
- (1 + b_j) \log \left( \frac{y}{p_j} \right).
\]

Now \((1 + b_i)\) and \((1 + b_j)\) are slope coefficients in the multiple regression of the dependent variable \((\log x_i - \log x_j)\) on the independent variables \(\log (y/p_i)\) and \(\log (y/p_j)\). From the estimate of each slope coefficient estimates of \(b_i\) and \(b_j\) can be computed. Is it possible to estimate \(a_i\) and \(a_j\)?

A final word has to be said about the interest of the indirect as compared with the direct utility function. We saw that they are equivalent representations of the underlying preference ordering and that maximization of the latter and minimization of the former leads to the same observable behaviour in the market, i.e. to the same demand equations. In Houthakker's words (1960b, p. 245): 'While the direct utility function probably has greater intuitive appeal, the indirect utility function is not without its claims to interest also, for it is the foundation of 'constant-utility' index numbers of the cost of living. If we try to determine what change in income is necessary to compensate for a given change in prices (to mention one of the problems to which such index numbers can be applied) we are in effect trying to keep the indirect utility function constant'. Chapter V will make this clear.