Demand functions: particular restrictions

In this chapter, our efforts concentrate on the specification of the utility function. Our objective is to derive specifications leading to interesting and operational restrictions, capable of moulding the available statistical data in a form that can easily be tested by current econometric methods and therefore rejected by the same methods. A first specification that comes to mind, because it is the first step to the analysis of the structure of human wants and has been widely used in the empirical work of the sixties, is that of an additive utility function.

3.1. Additive utility functions

Up to now, we have left the utility function entirely unspecified in the theoretical exposition. (Particular specifications appeared only in the exercises.) The utility function was therefore written as \( u = f(x_1, x_2, \ldots, x_n) \).

To introduce the (strong) assumption that the utility provided by the consumption of one good is not influenced by (independent of) the consumption of any other good, we write the direct utility function as

\[
u = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n) \tag{3.1}
\]

where \( f_i \) designates a function peculiar to commodity \( i \) (not to be confused with a partial derivative of \( f \)). A function obeying (3.1) is called 'additive'. However, any monotonically increasing transformation \( F(u) \) also represents the underlying preference ordering. We therefore say that a preference ordering, represented by a utility function \( u = f(x_1, \ldots, x_n) \),

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1 The basic reference is Houthakker (1960b).
is additive if there exists a differentiable function $F$, $F' > 0$, and $n$ functions $f_i(x_i)$, such that

$$F[f(x_1, \ldots, x_n)] = \sum_i f_i(x_i) \quad (i = 1, \ldots, n), \tag{3.1a}$$

i.e. if one member of the class of utility functions representing the preference ordering is additive.

It can easily be seen that (3.1) implies independence of the marginal utility of good $i$ from the consumption of any other good. As $\frac{\partial u}{\partial x_i} = \frac{df_i}{dx_i}$,

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad (i \neq j). \tag{3.2}$$

**EXERCISES**

3.1. (a) For what sort of commodities can the additivity assumption be considered as a good approximation of reality?
(b) Is any of the utility functions listed in exercise 1.7 additive?
(c) What about the Stone-Geary utility function introduced in exercise 1.14?

*Answer:* (a) The additivity assumption is defensible if the arguments of the utility function are taken to be broad aggregates of goods such as 'food', 'clothing', 'housing' rather than individual commodities. It is precisely for these aggregates that statistical data can be found in the national accounts.

3.2. In applied work, one often encounters the quadratic utility function. For two goods, it is written as

$$u = a_1 x_1 + a_2 x_2 + \frac{1}{2}(a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2).$$

(a) What does the equation look like in algebraic and in matrix notation for $n$ goods?
(b) How can it be made additive?
(c) What is the typical feature of marginal utilities?
(d) How can one ensure that marginal utilities are decreasing?
(e) Take the additive form of the quadratic utility function. Assume that prices are constant and equal to 1 and $n = 2$ (to simplify the arithmetic). Maximize subject to the budget constraint and solve for the demand equations. You should discover that these
Additive utility functions

equations (called ‘Engel curves’ because the quantities demanded are functions of income only) are linear.

Answer: (a) \[ u = \sum_i a_i x_i + \frac{1}{2} \sum_i \sum_j a_{ij} x_i x_j \]

and

\[ u = a'x + \frac{1}{2} x'Ax \]

where \( x \) is the column vector \([x_1, x_2, \ldots, x_n] \), \( a \) is the column vector of constants and \( A \) is a symmetric matrix of constants with the elements \( a_{ii} \) on its main diagonal and the \( a_{ij} \)'s as off-diagonal elements \( (a_{ij} = a_{ji}) \).

(b) Put all \( a_{ij} = 0 \) \((i \neq j)\). Then the matrix \( A \) is diagonal.

(c) Marginal utilities are linear and can become negative.

(d) Impose \[ \frac{\partial^2 u}{\partial x_i^2} = a_{ii} < 0. \]

(e) I find the system

\[ x_1 = \frac{a_2 - a_1}{a_{11} + a_{22}} + \frac{a_{22}}{a_{11} + a_{22}} y \]

\[ x_2 = \frac{a_1 - a_2}{a_{11} + a_{22}} + \frac{a_{11}}{a_{11} + a_{22}} y. \]

Adding up these two linear Engel curves, I find that the two intercepts sum to zero (the first is the negative of the second) and that the slopes sum to one. Why should this be?

It is worth noting that the early marginalist writers worked with additive utility functions. Because of the impact of ordinalism, these functions disappeared from the scene for many years (to reappear in the sixties, only, with the development of applied econometric work in the field). Additivity, defined as above, is indeed a cardinal property. The reader who has gone carefully through the material of Section 1.3 should have no difficulty to prove this.

Exercise

3.3. (a) Show that the condition (3.2) is invariant under linear transformations (of the utility function) only.

(b) Illustrate (a) by comparing the second order partial cross derivatives computed in exercise 1.10.
\textit{Answer:} (a) For \( v = F(u) \) with \( F' > 0 \), we have
\[
\frac{\partial^2 v}{\partial x_i \partial x_j} = F' \frac{\partial^2 u}{\partial x_i \partial x_j} + F'' \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}
\]
which is zero only if \( F'' = 0 \) (\( \partial u/\partial x_i \) and \( \partial u/\partial x_j \) being positive).

The additivity assumption is a first step towards a better understanding of the structure of preferences. It is a progress, in the sense that it allows us to take the independence of certain aggregates (groups of commodities) into account. But it is unfortunate in having additive utility functions defined in a cardinal way. Suppose, indeed, that a preference ordering is additive. It is a pity to have its representation cast in a form that is not invariant under non-linear transformations.

One possible attitude is to say that this fact should not prevent us from analysing and testing its empirical implications, as it is important to get a better understanding of the structure of preferences (which a purist ordinal approach would prevent us from analysing). This is the current attitude and we will adopt it in the chapters devoted to empirical studies.

A further possibility is to reject the definition of additivity in terms of the second order cross-partial derivatives of the utility function (equation (3.2)), and to define additivity in terms of the particular restriction which it implies. This restriction is derived below in equation (3.13) and is cast in terms of price and income derivatives of the demand equations, which are invariant.

Another more courageous attitude is to try to redefine independence (together with the other possible relationships among goods, which are substitutability and complementarity) in ordinal terms, exploiting the groupwise structure of preferences. Barten (1971) has done original work in this direction and we will try to get the flavour of it in Section 3.4.

For the time being, we are interested in the empirical implications of additivity, i.e. in the restrictions on the derivatives of the demand equations that result from direct additivity. To simplify the notation, we will analyse a two-commodity case, and generalize while proceeding.

Maximization of \( u = f_1(x_1) + f_2(x_2) \) subject to the budget constraint gives the first-order conditions
\begin{align*}
f_1' &= \lambda p_1, \\
f_2' &= \lambda p_2, \\
p_1 x_1 + p_2 x_2 &= y.
\end{align*}
(3.3)
Additive utility functions

where $f'_i = \partial f_i / \partial x_i$. Differentiation of (3.3) with respect to $y$ leads to

\[
\begin{align*}
 f_1^* \frac{\partial x_1}{\partial y} &= p_1 \frac{\partial \lambda}{\partial y} \\
 f_2^* \frac{\partial x_2}{\partial y} &= p_2 \frac{\partial \lambda}{\partial y} \\
 p_1 \frac{\partial x_1}{\partial y} + p_2 \frac{\partial x_2}{\partial y} &= 1
\end{align*}
\]  

(3.4)

as the cross-derivatives vanish. System (3.4) implies

\[
 p_1 \left( \frac{p_1}{f_1^*} \right) + p_2 \left( \frac{p_2}{f_2^*} \right) = \frac{1}{\partial \lambda / \partial y}
\]  

(3.5)

Differentiation of (3.3) with respect to $p_1$ gives

\[
\begin{align*}
 f_1^* \frac{\partial x_1}{\partial p_1} &= \lambda + p_1 \frac{\partial \lambda}{\partial p_1} \\
 f_2^* \frac{\partial x_2}{\partial p_1} &= p_2 \frac{\partial \lambda}{\partial p_1} \\
 x_1 + p_1 \frac{\partial x_1}{\partial p_1} + p_2 \frac{\partial x_2}{\partial p_1} &= 0
\end{align*}
\]  

(3.6)

or, after some manipulation,

\[
 p_1 \left( \frac{p_1}{f_1^*} \right) + p_2 \left( \frac{p_2}{f_2^*} \right) = \frac{-x_1 - p_1(\lambda f''_1)}{\partial \lambda / \partial p_1}
\]  

(3.7)

Similarly, differentiation of (3.3) with respect to $p_2$ gives

\[
 p_2 \left( \frac{p_2}{f_2^*} \right) + p_1 \left( \frac{p_1}{f_1^*} \right) = \frac{-x_2 - p_2(\lambda f''_2)}{\partial \lambda / \partial p_2}
\]  

(3.8)

Comparing (3.5), (3.7) and (3.8), we see that

\[
 \frac{1}{\partial \lambda / \partial y} = \frac{-x_1 - p_1(\lambda f''_1)}{\partial \lambda / \partial p_1} = \frac{-x_2 - p_2(\lambda f''_2)}{\partial \lambda / \partial p_2}
\]

(3.9)

\[
 = \frac{-x_k - p_k(\lambda f''_k)}{\partial \lambda / \partial p_k} \quad (k = 1, \ldots, n)
\]
Demand functions: particular restrictions

or

\[
\frac{\partial \lambda}{\partial p_k} = \frac{\partial \lambda}{\partial y} \left( -x_k - \frac{\lambda}{f_i} p_k \right)
= \frac{\partial \lambda}{\partial y} \left( -x_k - \frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_k}{\partial y} \right)
\]

(3.10)

using (3.4). From (3.6) we know that \( \partial x_i/\partial p_k = (p_i/f_i)(\partial \lambda/\partial p_k) \). Therefore

\[
\frac{\partial x_i}{\partial p_k} = -\frac{p_i}{f_i} \frac{\partial \lambda}{\partial y} \left( x_k + \frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_k}{\partial y} \right)
= -\frac{\partial x_i}{\partial y} \left( x_k + \frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_k}{\partial y} \right) \quad (i \neq k).
\]

(3.11)

Clearly, we also have

\[
\frac{\partial x_j}{\partial p_k} = -\frac{\partial x_j}{\partial y} \left( x_k + \frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_k}{\partial y} \right) \quad (j \neq k).
\]

so that

\[
\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{\partial x_i/\partial y}{\partial x_j/\partial y} \quad (i \neq k; j \neq k).
\]

(3.12)

This is the particular restriction we were looking for: the direct utility function can be written in the additive form if and only if\(^2\) the cross-price derivatives are proportional to the income derivatives. To understand what this means, write (3.12) as

\[
\frac{\partial x_i}{\partial p_k} = \mu \frac{\partial x_i}{\partial y} \quad \text{with} \quad \mu = \frac{\partial x_i/\partial p_k}{\partial x_j/\partial y}, \quad i \neq k,
\]

(3.13)

and suppose \( i \) designates food and \( k \) designates housing. The restriction says that the change in the demand for food induced by a change in the price of housing is proportional to the change in the demand for food induced by a change in income. The factor of proportionality \( \mu \) does not depend on the good (food) whose quantity response we are considering (but it does depend on the good (housing) whose price has changed). If \( i \) were to designate another good, say clothing, the factor of proportionality would be the same, as long as it is the price of housing that is changing.

\(^2\) For a proof of sufficiency, see Houthakker (1960b).
Notice also that we can express the same restriction in terms of the cross-substitution effect. In general, the cross substitution effect is equal (see exercise 2.8) to

$$k_{ij} = \frac{\partial x_i}{\partial p_j} + x_i \frac{\partial x_j}{\partial y}.$$ 

Using (3.11), we find that in the case of direct additivity it reduces to

$$k_{ij} = -\frac{\partial x_i}{\partial y} \left( x_j + \frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_j}{\partial y} \right) + x_j \frac{\partial x_i}{\partial y}$$

$$= -\frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_i}{\partial y} \frac{\partial x_j}{\partial y} \quad (i \neq j)$$

which we identified in Section 2.4 as the 'general' cross substitution effect. The specific cross-substitution effect is zero.

This is an important result. First of all, it shows that the independence of the marginal utility of good $i$ from the consumption of any other good $j$ does not imply that a change of the price of good $j$ leaves the demand of good $i$ unaffected: the cross-substitution effect does not vanish! Even under additivity, a change in the price of any other good will affect the demand of good $i$. Why is this? Because all goods 'compete for the consumer's dollar', to put it in simple terms. In other words, there always remains an overall effect, which we call the general substitution effect, and which is proportional to the income derivatives $\partial x_i/\partial y$ and $\partial x_j/\partial y$. The factor of proportionality is a function of the marginal utility of money and its response to a change in income.

On the other hand, we established in Chapter III that the Slutsky equation is invariant under monotonic increasing transformations of the utility function. This means, in particular, that the substitution effects are invariant. All the utility functions belonging to the same class as a given additive utility function (i.e. including the non-linear transformations of it which are non-additive) have therefore the same cross-substitution effects (for given $i$, $j$). In the case of the additive function, the 'total' effect reduces to one of its components, which is not invariant (as the ratio $\lambda/(\partial \lambda/\partial y)$ is invariant under linear transformations only).

When we transform the additive member of the class non-linearly, the general substitution effect changes, but simultaneously the specific effect becomes non-zero in such a way that the sum of the general and the specific effect, i.e. the ('total') substitution effect, remains unaltered.
EXERCISES
3.4. (a) Verify that the cross-substitution effects (for given \( i, j \)) are the same for the utility functions (1), (2), (4), (5) and (6) of exercise 1.7.
(b) Check that the last of these utility functions, namely \( u = \log (x_i x_j) \), is the only one to have cross-substitution effects obeying equation (3.14).

3.5. (a) Prove the following proposition: when the marginal utility of each good is a function of its own quantity only and decreases when the consumption of the good increases, then consumption of each good increases with income.
(b) What is the sign of the cross-substitution effect in the case described under (a)?
*Hint:* (a) Prove that \( \partial x_i / \partial y \) is positive, starting from the system (2.8) and using Cramer's rule, for \( n = 2 \).

Up to now, we discussed direct additivity. When it is the *indirect* utility function that is additive, the resulting particular restriction has been shown by Houthakker (1960b) to be

\[
\frac{\partial x_i / \partial p_k}{x_i} = \frac{\partial x_j / \partial p_k}{x_j}, \quad (i \neq k, j \neq k)
\]  
(3.15)

which is quite different from restriction (3.12). Here, the cross-price derivatives are proportional to the quantities affected. Indirect additivity has other implications than direct additivity! (3.15) can in turn be written as

\[
\frac{\partial x_i}{\partial p_k} \frac{p_k}{x_i} = \frac{\partial x_j}{\partial p_k} \frac{p_k}{x_j}
\]  
(3.16)

so that all cross-price elasticities with respect to \( p_k \) are the same.

What if the direct and the corresponding indirect utility functions are both additive? Combining (3.12) and (3.15), one finds

\[
\frac{\partial x_i / \partial p_k}{\partial x_j / \partial p_k} = \frac{\partial x_i / \partial y}{\partial x_j / \partial y} = \frac{x_i}{x_j}
\]

or

\[
\frac{\partial x_i}{\partial y} \frac{y}{x_i} = \frac{\partial x_j}{\partial y} \frac{y}{x_j}
\]  
(3.17)
that is to say, all elasticities with respect to total expenditure are equal and hence unitary\(^3\). As will become clear in the next chapter, this implies that all Engel curves are straight lines through the origin: at all income levels, a constant proportion of total expenditures is allocated to each commodity, which is in contradiction with all empirical findings. Fortunately, the indirect utility function corresponding to an additive direct utility function is not, in general, additive. This is fortunate, because otherwise additive utility functions would be entirely unrealistic and not suitable for empirical work.

**Exercises**

3.6. (a) How would you write the additive indirect utility function, taking account of the fact that it is homogeneous of degree zero in income and prices?
(b) In earlier exercises we came across an additive direct utility function whose corresponding indirect utility function is also additive. Which one is it? Convince yourself that all income elasticities are unitary.

*Answer:* (a) \(u^* = f_1^* \left( \frac{y}{p_1} \right) + f_2^* \left( \frac{y}{p_2} \right) + \ldots + f_n^* \left( \frac{y}{p_n} \right)\).
(b) see exercise 1.19, utility function (6).

3.7. The preceding exercise made it clear that, while direct additivity is defined in terms of the \(x_i\)'s, indirect additivity is defined in terms of income and prices. The Stone-Geary utility function and the corresponding indirect utility function are not additive in these variables. You will have noticed, however, that the preference orderings described by these utility functions are additive in other variables. Which are these? Show that restriction (3.17) applies to them.

*Hint:* see exercises 1.14 and 1.26.

The reader who has gone through exercises 3.6, 1.19 and 2.9, will have discovered that the class of utility function to which \(u = \sum_i \log x_i\) belongs leads to demand functions with not only unitary income

\(^3\) For a further discussion of this restriction, see Samuelson (1965). Houthakker (1960) and Section 3.5.
elasticiies, but also *unitary own price elasticiies*\(^4\) and zero *cross-price elasticiies*. This is an implication of 'simultaneous' direct and indirect additivity, as has been shown by Samuelson (1965).\(^5\)

In fact, each of these properties implies any of the other, as the demand functions must be of the form \(x_i = k_i(y/p_i)\), where \(k_i\) is a constant, in order to have \(p_i x_i = k_i y\). Once this is realized, the reader should conclude that something is wrong with the derivation of (3.17), which is based on a combination of (3.12) and (3.15). Indeed, knowing that simultaneous direct and indirect additivity implies zero *cross-price elasticiies*, the ratios

\[
\frac{\partial x_i}{\partial p_k} = 0 \\
\frac{\partial x_j}{\partial p_k} = 0
\]

become indeterminate forms and one cannot legitimately manipulate their equality as if they were regular numbers, as was pointed out by Samuelson (1969). A correct proof of (3.17) should proceed otherwise, by using the duality properties of direct and indirect utility functions. It is beyond the scope of this book to develop these duality relations.

The utility functions which are directly and indirectly additive are also 'homothetic' and will be further discussed in the final section of this chapter.

### 3.2. Separable utility functions

The aggregates that appear as arguments in additive utility functions are groups of commodities. A natural question is to ask under what conditions the arguments of the utility functions may be aggregated. In particular, is not additivity too strong a condition, in the sense that a grouping is possible without supposing that the groups are independent? The answer is: yes.

It is intuitively clear that what we would like to do is to partition the consumption set into subsets which would include commodities that

\(\text{\footnotesize\(^4\) To be entirely correct, one should draw attention to the existence of an exceptional case, discovered by Hicks (1969) and further analysed by Samuelson (1969), where at most one commodity may have a non-unitary price elasticity.}

\(\text{\footnotesize\(^5\) The adjective 'simultaneous' is used here to indicate the case where the indirect utility function corresponding to an additive direct utility function is additive. We have 'non-simultaneous' direct and indirect additivity when the corresponding indirect utility function is non-additive but can be transformed into an additive one.}\)
are closer substitutes or complements to each other than to members of other subsets. (Independence from one subset to another would not be required.) Instead of writing, say

\[ u = f(x_1, x_2, x_3, x_4, x_5) \]

we would like to group the variables in the function to make it expressible as, say

\[ u = F(A, B) \]

where

\[ A = f_A(x_1, x_2) \]
\[ B = f_B(x_3, x_4, x_5). \]

Commodities 1 and 2 would, in this example, belong to group \( A \) and commodities 3, 4 and 5 to group \( B \). The utility function would then be 'separable' (in two groups, in the example) without necessarily being additive. The term 'expressible' indicates an important condition: we want the value of \( u \) to be the same with or without grouping, i.e. whether \( u \) is expressed as a function of all elementary variables or as a function of the groups.

Under what conditions can this be done? Following Green (1964, p. 11), we start from a simple numerical example, in which the consumer is supposed to consume the following three commodities: commodity \( A \) (food), commodity \( D_1 \) (housing) and \( D_2 \) (automobiles). We seek the conditions under which \( u = f(A, D_1, D_2) \) can be written as \( u = F(A, D) \) where \( D \) is a function of \( D_1 \) and \( D_2 \) and aggregates \( D_1 \) and \( D_2 \) into the group 'durable goods'.

Let us agree to represent two combination of \( D_1 \) and \( D_2 \), combined with a constant amount of \( A \), by the same value of \( D \) if and only if they provide the same satisfaction (when combined with the given amount of \( A \)). This is in keeping with the requirement that a grouping of the variables should not modify \( u \). Thus, we shall say that \( D(20, 5) = D(18, 6) \) from the point of view of utility, if the following figures are given:

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>20</td>
<td>5</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>18</td>
<td>6</td>
<td>1.000</td>
<td></td>
</tr>
</tbody>
</table>
Suppose the quantity of $A$ consumed doubles. If we find

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>20</td>
<td>5</td>
<td>1.800</td>
</tr>
<tr>
<td>200</td>
<td>18</td>
<td>6</td>
<td>1.700</td>
</tr>
</tbody>
</table>

our convention obliges us to say that $D$ (20, 5) $\neq$ $D$ (18, 6). It is then impossible to say anything more about $D$: there is no basis for grouping $D_1$ and $D_2$ into 'durables'. But if the two combinations of $D_1$ and $D_2$, when consumed with the new (or any other) quantity of $A$, still provide the same utility, then we are sure (and only then) that the durables are separable from food. This is the case if the following figures hold:

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>20</td>
<td>5</td>
<td>1.800</td>
</tr>
<tr>
<td>200</td>
<td>18</td>
<td>6</td>
<td>1.800</td>
</tr>
</tbody>
</table>

Now, to say that the two combinations of $D_1$ and $D_2$ provide the same utility, whatever the consumption of $A$, is to say that the marginal rate of substitution of one durable for another ($20 - 18 = 2D_1$ for $6 - 5 = 1D_2$) is independent of $A$. Generalizing, this leads to the criterion that it is a necessary and sufficient condition for a function to be separable, that the marginal rate of substitution between any two variables belonging to the same group be independent of the value of any variable in any other group.

This theorem is due to Leontief (1947). To prove necessity, let the $n$ commodities be partitioned in $m$ groups, and let there be $n_r$ ($r = 1, \ldots, m$) commodities in each group (\(n = \sum_{r=1}^{m} n_r\)). The utility function

$$u = f(x_{11}, \ldots, x_{1n_{1}}, \ldots, x_{r1}, \ldots, x_{rn}, \ldots, x_{m1}, \ldots, x_{mn})$$

(3.18)

is to be expressed in the form

$$u = F[f_1(x_1), f_2(x_2), \ldots, f_r(x_r), \ldots, f_m(x_m)]$$

(3.19)

where each $f_r$ is a 'branch'-utility function. Each $x_r$ is in turn a function of $x_{r1}, x_{r2}, \ldots, x_{rn}$. 


Separable utility functions

We want to show that (3.18) can be expressed as (3.19) if and only if

\[
\frac{\partial}{\partial x_{qk}} \left( \frac{\partial f}{\partial x_{ri}} \right) = 0 \tag{3.20}
\]

for all \( q, r, i, j, k \) \((q, r = 1, \ldots, m; q \neq r; k = 1, \ldots, n_q; i, j = 1, \ldots, n_r)\).

We start from the condition that the grouping should not affect \( u \), and that therefore

\[
df = \sum_r \sum_j \frac{\partial f}{\partial x_{ri}} dx_{ri} = dF = \sum_r \frac{\partial F}{\partial x_r} dx_r = \sum_r \sum_j \frac{\partial F}{\partial x_r} \frac{\partial f}{\partial x_{ri}} dx_{ri},
\]

For any group \( r \), and any members \( i \) and \( j \), it follows that

\[
\frac{\partial f}{\partial x_{ri}} = \frac{\partial F}{\partial x_r} \frac{\partial f}{\partial x_{ri}} \quad \text{and} \quad \frac{\partial f}{\partial x_{ri}} = \frac{\partial F}{\partial x_r} \frac{\partial f}{\partial x_{ri}}
\]

so that

\[
\frac{\partial f}{\partial x_{ri}} \frac{\partial f}{\partial x_{ri}} = \frac{\partial f}{\partial x_{ri}} \frac{\partial f}{\partial x_{ri}}
\]

But since \( f_r \) is a function only of the members of group \( r \), the same is true for the ratio of its derivatives and for the marginal rate of substitution to which this ratio is equal. Q.E.D.

The proof of sufficiency is rather cumbersome and, in order to save space, will not be given here (see Green, 1964, p. 13–15).

In the exposition above, goods \( i \) and \( j \) belong to the same group \( r \), while good \( k \) belongs to a different group \( q \). It is clear that one could define another type of separability, in which \( i, j \) and \( k \) belong each to a different group. This is called ‘strong separability’. Then, the partition is such that the marginal rate of substitution between any two goods belonging to two different groups is independent of the consumption of any good in any third group. As a result, the marginal utility of a commodity in one group is independent of the consumption of any good in any other group, which is a very strong condition indeed. Strong separability thus implies independence between groups or ‘groupwise independence’. (By contrast, functional separability as defined by condition (3.20) has been called ‘weak separability’.) A strongly separable utility function is written

\[
u = f_1(x_1) + f_2(x_2) + \cdots + f_r(x_r) + \cdots + f_m(x_m) \tag{3.21}\]

where each \( f_r \) is a branch utility function.
EXERCISES

3.8. Utilize one of the algebraically specified utility functions, introduced in earlier exercises, to construct:
(a) a weakly separable utility function;
(b) a strongly separable utility function.

*Answer:* Why not take the quadratic utility function? (You might as well try the Stone-Geary function.) To get a clear distinction between the two cases, we have to introduce at least three commodities, one of which is an aggregate. Let \( x_i = b_1 x_{1i} + b_2 x_{12} \), i.e. commodity 1 is a linear combination of two more disaggregated commodities.

(a) Then \( u = a_1 x_1 + a_2 x_2 + a_3 x_3 + \frac{1}{2}(a_{11} x_1^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + 2a_{23} x_2 x_3 + a_{22} x_2^2 + a_{33} x_3^2) \) is weakly separable. Indeed

\[
\frac{\partial u}{\partial x_{11}} \frac{\partial u}{\partial x_{12}} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial x_{11}} \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial x_{12}} = \frac{b_1}{b_2}
\]

is independent of \( x_2 \) and \( x_3 \). Notice that

\[
\frac{\partial u}{\partial x_{11}} = (a_1 + a_{11} x_1 + a_{12} x_2 + a_{13} x_3) b_1
\]

is not independent of \( x_2 \) or \( x_3 \). The same is true for \( \partial u/\partial x_{12} \).

(b) By putting \( a_{ij} = 0 \) \((i \neq j)\), we obtain a strongly separable utility function. Then

\[
\frac{\partial u}{\partial x_{11}} = \frac{(a_1 + a_{11} x_1) b_1}{(a_2 + a_{22} x_2)}
\]

is independent of \( x_3 \). Now \( \partial u/\partial x_{11} \) is independent of \( x_2 \) and \( x_3 \).

3.9. Is strong separability synonymous with additivity?

*Answer:* Formally, strong separability implies additivity between groups. It is only when each group comprises only one commodity that strong separability reduces to additivity. From an economic point of view, however, additivity is defensible only if the arguments of the utility function are broad aggregates (see exercise 1.19). In applied work, strong separability, groupwise independence and additivity therefore designate one and the same situation.
3.10. Specify algebraically a utility function that is strongly separable \((m = 3)\) but with a first branch that is weakly separable \((n_1 = 2)\).

*Hint:* This can be obtained by adding an additive quadratic utility function to a non-additive one.

Weak separability is of great interest to the economist as a prerequisite for the consistency of the two-stage maximization procedure discussed by Strotz\(^6\). The basic idea is appealing and seems highly plausible. Households are supposed to proceed in two steps, the first being an optimal allocation of income among broad commodity groups (the \(m\) groups in equation (3.19), say) corresponding to 'branches' of the utility function. There is thus a budget allotment \(y_r (\sum y_r = y)\) to each branch.

The second step implies the optimal spending of each budget allotment in its branch, with no further reference to purchases in other branches. This amounts to maximizing each branch utility function subject to the budget constraint

\[
\sum_{j} x_{rj} p_{rj} = y_r \quad (j = 1, \ldots, n_r).
\]

The (weakly) separable utility function appears, in the terminology introduced by Strotz, as a *utility tree*, with branches corresponding to \(f_1, f_2, \ldots, f_m\).

The time has come to ask ourselves what are the implications for observed consumer behaviour of the separability hypothesis. These implications can be derived in a simple way with the help of so-called 'conditional demand functions', which we introduce in the next section, following Pollak (1969) and especially (1971a). These functions have the additional advantage of placing emphasis on the implications of separability for the demand functions themselves, by facilitating the derivation of restrictions not only on the derivatives of the demand equations, but also on the number of variables that appear as arguments in the demand equations.

\(^6\) In his 1957 article, Strotz defined the utility function as being weakly separable. Gorman (1959a, b) tried to show – and Strotz (1959) subsequently agreed with this – that the two-stage procedure has to be based, for three or more groups, on a strongly separable function. Recently, however, Blackorby et al. (1970) made it clear that the Strotz-Gorman conditions refer only to the existence of group price indices and showed that the budgeting procedure described by Strotz is consistent if, and only if, the utility function is weakly separable.
3.3. Conditional demand functions and the implications of separable utility

Let us partition the set of all commodities into two subsets, \( \theta \) and \( \bar{\theta} \). The individual's consumption of the goods in \( \bar{\theta} \) has been determined before he enters the market: these goods have been 'preallocated'. We assume that the consumer is not allowed to sell any of his allotment of the preallocated goods, and that he cannot buy more of them. The goods in \( \theta \) are available on the market. Total expenditure on the goods available on the market is denoted by \( y_\theta \). The consumer is supposed to maximize \( u = f(x_1, \ldots, x_n) \) subject to the constraint

\[
\sum_{i \in \theta} p_i x_i = y_\theta
\]  

(3.23)

and the additional constraints

\[
x_k = \bar{x}_k \quad (k \in \bar{\theta}).
\]  

(3.24)

Hence, his demand for the goods available on the market depends on the prices of these goods \( (P_\theta) \), total expenditure on them \( (y_\theta) \), and his allotment of the preallocated goods (\( \bar{x} \) say). Then the conditional demand function for the \( i \)th good is

\[
x_i = \psi_i(P_\theta, y_\theta, \bar{x}) \quad (i \in \theta).
\]  

(3.25)

Suppose now that the allotment of the preallocated goods is precisely equal to the amounts he would have purchased in the absence of a pre-allocation, i.e. to the amounts determined by the ordinary demand equations \( x_k = \phi_k(p_1, \ldots, p_n, y) \), and that \( y_\theta = y - \sum_k p_k \theta_k \) (\( k \in \bar{\theta} \)). Then the consumer will purchase the same quantities of each of the goods available (in \( \theta \)) as in the absence of a pre-allocation. That is

\[
\phi_i(p_1, \ldots, p_n, y) = \psi_i(P_\theta, y_\theta, \Phi_\theta)
\]  

(3.26)

where \( \Phi_\theta \) designates the vector of ordinary demand equations for the preallocated goods.

Coming back to the case where the utility function is a tree, i.e. is weakly separable according to (3.19), we assume that the goods in one branch (the branch 'food') are available on the market, while all other
Conditional demand functions

goods are preallocated. The conditional demand functions are determined by maximizing (3.19) subject to

$$\sum_{j=1}^{n_r} x_{rq} p_{rj} = y_r$$

which is the constraint (3.22), \( r \) being the branch ('food') in which the goods are available on the market, and the additional constraints

$$x_{rq} = \bar{x}_{qr}, \quad (qk \in \theta). \quad (3.27)$$

If we absorb the constraints (3.27) into (3.19), we obtain

$$u = F[f_1(\bar{x}_1), f_2(\bar{x}_2), \ldots, f_r(x_r), \ldots, f_m(\bar{x}_m)]. \quad (3.28)$$

Our problem now reduces to the constrained maximization of (3.28), which itself reduces to the constrained maximization of the branch utility function \( f_r(x_r) \) so that the utility maximizing values of \( (x_{r1}, x_{r2}, \ldots, x_{rn_r}) \) are independent of the levels of the preallocated goods. Hence, the conditional demand functions for the goods in \( \theta \), i.e. in branch \( r \), are of the form

$$\psi_{ri}(P, y_r, \bar{x}) = \psi_{ri}(P, y_r),$$

while, using (3.26), the ordinary demand functions for the goods in branch \( r \) are such that

$$x_{ri} = \phi_{ri}(P, y_r) = \psi_{ri}(P, y_r) \quad (3.29)$$

where \( P \) is the vector of all prices \( (p_{11}, \ldots, p_{1m_1}, \ldots, p_{r1}, \ldots, p_{rn_r}, \ldots, p_{m_1}, \ldots, p_{m_m}) \).

Equation (3.29) is important. It is a restriction on the nature of the arguments that appear in the demand equations: (3.29) says that the demand for a commodity in a branch can be expressed as a function of the prices in and the budget allotment to that branch.

We do not say that the quantities demanded in one branch are independent of the prices of commodities in other branches or of total expenditures. What we say is that total income and the prices of goods outside the branch enter the demand functions for goods in the branch only through their effect on \( y_r \), the budget allotment to that branch. And that, when the budget allotment to the branch is known, we can ignore the prices of goods outside the branch.

This restriction is very useful in applied work. To the extent that the hypothesis of a utility tree, and the corresponding assumption of multi-
Demand functions: particular restrictions

stage maximization, can be maintained, and if the branch structure of a specified utility function is known, then a subset of demand equations inside a branch can be estimated using only the prices of the goods in the branch and total expenditures on goods in the branch.

Exercises

3.11. Pollak (1971a) illustrates the preceding discussion by supposing that \( r = \text{food} \) and \( ri = \text{Swiss cheese} \).

(a) To what extent will a change in the price of shoes (= \( qk \), \( q \neq r \)) cause a change in Swiss cheese consumption?

(b) Suppose that a change in the price of another non-food item, say, tennis balls, has the same effect on \( y_r \) as the change in the price of shoes. Will both price changes have the same effect on the consumption of Swiss cheese?

(c) What if a change in total income \( y \) has the same effect on \( y_r \) as the two price changes just considered? Will it also have the same effect on Swiss cheese consumption?

3.12. Suppose you have estimated a system of demand equations for the branch food. What additional information do you need to compute

(a) the cross-price elasticity between Swiss cheese and shoes?

(b) the elasticity of Swiss cheese with respect to total income \( y \)?

The implications of the utility tree hypothesis for the partial derivatives of the demand functions can now be derived without difficulty. Differentiating (3.29) with respect to \( p_{rk} \) and \( y \) we obtain

\[
\frac{\partial x_{ri}}{\partial p_{rk}} = \frac{\partial \psi_{ri}}{\partial y_r} \frac{\partial y_r}{\partial p_{rk}} \quad (q \neq r)
\]

\[
\frac{\partial x_{ri}}{\partial y} = \frac{\partial \psi_{ri}}{\partial y_r} \frac{\partial y_r}{\partial y} \quad (3.30)
\]

taking account of the fact that \( y_r \) is a function of all prices and of total income \( y \). That is, the change in the consumption of a good in branch \( r \) caused by a change in the price of a good in another branch is proportional to the change in the budget allotment of the branch caused by that price change. And similarly for a change in total income. (Notice that this restriction is implicitly contained in the answer to exercise 3.12.)
Conditional demand functions

On eliminating \( \partial \psi_{ri} / \partial y_r \) from (3.30) and supposing that \( \partial y_r / \partial y \neq 0 \) we can write

\[
\frac{\partial x_{ri}}{\partial p_k} = \frac{\partial y_r / \partial p_k}{\partial y} \frac{\partial x_{ri}}{\partial y}
\]

\( = \mu_r \frac{\partial x_{ri}}{\partial y} \) \hspace{1cm} (3.31)

That is, the change in the demand for Swiss cheese induced by a change in the price of shoes is proportional to the change in the demand for Swiss cheese induced by a change in income. The factor of proportionality is the same for all food items (but it does depend on the good (shoes) whose price has changed).

Introducing a second food item \( j \), we can express (3.31) as

\[
\frac{\partial x_{ri} / \partial p_k}{\partial x_{rj} / \partial p_k} = \frac{\partial x_{ri} / \partial y}{\partial x_{rj} / \partial y} \quad (q \neq r)
\]

(3.32)

Exercise

3.13. Compare equations (3.31) and (3.13) on the one hand, and (3.32) and (3.12) on the other hand. Notice the similarities and the differences.

We now turn to the case of strong separability or 'groupwise independence' (also called 'block additivity') defined in equation (3.21). It implies weak separability: as the marginal rate of substitution between any two goods belonging to two different groups is independent of the consumption of any good in any third group, the marginal rate of substitution between two goods belonging to the same group depends only on the goods in that group. The restriction derived above therefore applies directly to groupwise independence: the demand for a particular commodity can be written as a function of the prices in the branch and of the budget allotment to the branch to which it belongs. As groupwise independence is a stronger condition than weak separability, it must have additional implications.

Pollak (1971, p. 248) derives these from the following consideration: 'If a utility function is a tree with \( m \) branches, in general, we cannot combine two branches into a single branch. For example, if 'food' and 'recreation' are two branches of a tree, it is not in general true that the demand for Swiss cheese can be written as a function of food prices,
recreation prices, and total expenditure on food and recreation. But if the utility function is block additive, it is always permissible to treat the goods in two (or more) blocks as a single block. [...] More generally, if a block additive utility function has \( m \) blocks, and if some of these blocks are combined to form \( m^* \) superblocks, \( m^* < m \), then the utility function is block additive in the superblocks. Hence, a block additive utility function with \( m \) blocks is a utility tree with \( m^* \) branches.

Suppose, then, than \( r \) is a group combining all blocks except one, which is group \( q \). The latter contains the preallocated goods, say all 'clothing'. Then the demand for Swiss cheese may be written as a function of the prices of all non-clothing goods and total expenditure on all goods other than clothing. Restriction (3.30) applies, together with (3.31) and (3.32). But the interpretation is different: as \( r \) now indicates any 'superblock', the group to which good \( i \) belongs is irrelevant. Equation (3.31) should therefore be rewritten as

\[
\frac{\partial x_{ri}}{\partial p_{rq}} = \mu \frac{\partial x_{ri}}{\partial y},
\]

(3.33)

where the factor of proportionality has no subscript, to indicate that it is independent of the good whose quantity response we are considering.

It now suffices for us to imagine that each group contains only one commodity ('food', 'clothing', 'recreation', 'housing', ...), to be back in the case considered in Section 3.1. The results obtained for the case of block additivity carry over directly. In particular, we may use any partition of the goods which is convenient. For example, we can write the demand for food as a function of the price of food, the price of clothing, and total expenditure on food and clothing. Or, if we prefer, we can write the demand for food as a function of all goods except clothing and total expenditure on all goods other than clothing. As for the restriction on the price and income derivatives, it can now be derived without effort using conditional demand derivatives. The reader is invited to work this out for himself in the following exercise.

**Exercise**

3.14. Derive equations (3.13) and (3.12) by differentiating a conditional demand equation defined in the appropriate way. Give an economic interpretation of each of your results.

*Hint:* Suppose all goods are in \( \theta \), except good \( k \), which is pre-allocated.
It remains for us to express the restrictions on the derivatives, in the weak and strong separability cases, in terms of the cross-substitution effects $k_{ij}$, as we were able to do in equation (3.14) for the additivity case. Goldman and Uzawa (1964) have done pioneering work in this respect. As the matter is a difficult one, and the derivation is highly technical, the reader will have first to assimilate the material of the next section, at the end of which the said restrictions are derived.  

3.4 The structure of preferences: substitution, complementarity and independence

The hypothesis of independence between goods (whether particular commodities or groups of commodities) gives no more than a first approximation to the structure of preferences, even for very large aggregates. Inside these groups, complementarity and substitutability certainly exist. And it is highly likely that such relationships also exist between groups, at all levels of aggregation. This is at least the impression one gets after carrying out a test of the additivity hypothesis, as explained in Section 8.5 of Part II.

The hypothesis of weak separability seems therefore to give a more realistic description of the structure of preferences and to provide the appropriate framework to discuss complementarity and substitutability. It has the additional advantage of eliminating a number of paradoxes and contradictions that characterize the more traditional approaches. The old cardinal definition of, say, complementarity, is indeed different from the (by now traditional) ordinal definition. Furthermore, two commodities which appear as complements under one definition may appear as substitutes under another definition. Before presenting the new approach based on the separability hypothesis, we examine more closely the traditional definitions.

The cardinal definitions proceed in terms of second order cross-partial derivatives of the utility function. As we have seen in Section 3.1, independence between commodities $i$ and $j$ is defined by the fact that

\[
\frac{\partial^2 u}{\partial x_i \partial x_j} = 0.
\]

\footnote{See equations (3.45) and (3.46).}
Demand functions: particular restrictions

We may now add that, in the same spirit, complementarity implies

\[
\frac{\partial^2 u}{\partial x_i \partial x_j} > 0.
\]

Two commodities are complements if the increased consumption of \( j \) increases the marginal utility of \( i \) (and vice-versa). This definition probably corresponds to the reader's intuitive concept of complementarity. In a similar way, a negative second order cross-partial derivative defines substitutability.

However, the sign of these cross-partial derivatives is not invariant under monotonic increasing transformations of the utility functions, as we have demonstrated in Section 1.3 and re-emphasized in exercise 3.3. It is invariant under linear transformations only. That is why the above definitions are cardinal.

From an ordinal point of view, the demand equations have the advantage of being invariant under any monotonic increasing transformation (see exercises 1.11 and 1.15). As a consequence, the derivatives – and in particular the substitution effects – of the demand equations are invariant. Furthermore, the sign of the cross-substitution effect is not only invariant but also undetermined: it may be positive, negative or zero (see Section 2.3). This immediately suggests that one might use the sign of the cross-substitution effect \( k_{ij} \) and say that \( i \) and \( j \) are substitutes whenever \( k_{ij} \) is positive. Indeed, a compensated increase in the price of \( j \) (margarine) leads to an increase in the demand for \( i \) (butter).

We end up with the following invariant conditions:

\[
\begin{align*}
   k_{ij} > 0 & \text{ indicates substitutability} \\
   k_{ij} < 0 & \text{ indicates complementarity} \\
   k_{ij} = 0 & \text{ indicates independence.}
\end{align*}
\]

These are definitions suggested\(^8\) by Hicks (1936). They are to be found in most textbooks.

Notice that independence is \textit{not} defined by the fact that the demand for commodity \( i \) is independent of the price of commodity \( j \), i.e.

\[
\frac{\partial x_i}{\partial p_j} = 0 \quad (i \neq j)
\]

\(^8\) In fact, Hicks uses three different definitions of complementarity.
For $p_i$ to be the only price that appears in the demand equation of commodity $i$, it is indeed necessary that the cross-substitution effect be positive and equal in absolute value to the income effect (which is negative in the absence of inferior goods), so that their sum be equal to zero. We came across such a case in exercise 29. We then discovered that the utility function $u = \prod_i (x_i)$ belongs to the class of functions leading to demand equations with unitary own price and income elasticities, all cross-price elasticities being zero. According to the Hicksian definitions, all commodities would then appear as substitutes, as the cross-substitution effects are positive.

For some time, economists thought that this settled the matter. On second thoughts, they came to realize that the Hicksian definitions were not satisfactory.

First of all, they are biased in favour of substitutability. The use of the cross-substitution effect implies that all goods can be substitutes but not complements. To see this, recall that the homogeneity restriction implies

$$\frac{\partial x_i}{\partial y} + \sum_j p_j \frac{\partial x_i}{\partial p_j} = 0 \quad (i, j = 1, \ldots, n)$$

while the Slutsky equation gives

$$\frac{\partial x_i}{\partial p_j} = k_{ij} = \frac{\partial x_i}{\partial y} x_j.$$

Hence

$$\frac{\partial x_i}{\partial y} y + \sum_j p_j \left( k_{ij} - \frac{\partial x_i}{\partial y} x_j \right) = 0$$

$$\frac{\partial x_i}{\partial y} y + \sum_j p_j k_{ij} - \frac{\partial x_i}{\partial y} \sum_j p_j x_j = 0$$

or

$$\sum_j p_j k_{ij} = 0.$$  \hspace{1cm} (3.34)

As $k_{ii} < 0$, all $k_{ij}$ (for $i \neq j$) cannot be negative (complementarity). But all $k_{ij}$ could be positive. Furthermore, in the particular case of a directly additive utility
function, the substitution effect reduces to the 'general' substitution effect

\[ k_{ij} = -\frac{\lambda}{\partial \lambda/\partial y} \frac{\partial x_i}{\partial y} \frac{\partial x_j}{\partial y}. \]

On the assumption that marginal utilities are decreasing, all income derivatives are positive (see exercise 3.5), while \( \lambda \) is positive and \( \partial \lambda/\partial y \) is negative, so that \( k_{ij} \) is positive. In this case of independent marginal utilities, all goods are substitutes according to the Hicksian definitions. Clearly, this so-called substitutability has nothing in common with the preference relationships that are typical for the pairs of goods under consideration. It simply reflects the fact that all goods 'compete for the consumer's dollar'.

This result suggests another drawback of the Hicksian approach: it leads us away from the intuitive (cardinal) concepts to the point of giving contradictory and often confusing results. In the preceding paragraph, goods that are 'independent' (in a cardinal sense) appear as substitutes. In exercise 2.9, goods that are complements according to common sense (positive second order cross-partial derivatives of the utility function \( u = x_1 x_2 \)) are again gratified with positive cross-substitution effects (Hicksian substitutability).

In a recent paper, Barten (1971) has derived concepts that combine the intuitive appeal of the cardinal criteria with the ordinal character of the Hicksian definitions. To do this, Barten starts from the groupwise structure of preferences.

Consider a partition of the \( n \) commodities in \( m \) groups and let the utility function be a utility tree (or weakly separable) according to

\[ u = F[f_1(x_1), f_2(x_2), \ldots, f_r(x_r), \ldots, f_m(x_m)]. \]

Then

\[ \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial f_r} \frac{\partial f_r}{\partial x_i}, \quad \text{(3.35)} \]

for \( i \in r \), where \( r \) is any group. For \( i \in r, j \in q \) and \( r \neq q \), one has

\[
\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial f_r \partial f_q} \frac{\partial f_r}{\partial x_i} \frac{\partial f_q}{\partial x_j} \\
= \left( \frac{\partial^2 u}{\partial f_r \partial f_q} \right) \frac{\partial u}{\partial f_r} \frac{\partial u}{\partial f_q} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.
\]

\[
\text{(3.36)}
\]
The structure of preferences

At equilibrium, $\partial u/\partial x_i = \lambda p_i \ (i = 1, \ldots, n)$. Therefore

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \tau_{rq} p_i p_j \quad (3.37)$$

where

$$\tau_{rq} = \lambda^2 \frac{\partial^3 u}{\partial f_i \partial f_q} \left| \frac{\partial u}{\partial f_i} \frac{\partial u}{\partial f_q} \right| \cdot \frac{\partial f_i}{\partial f_q} .$$

Let us call $\tau_{rq}$ the interaction coefficient between groups $r$ and $q$. The sign of $\tau_{rq}$ is not invariant under non-linear (monotonic increasing) transformations of the utility function. However, the sign of the difference between two interaction coefficients is invariant! The proof will be given in a moment. We notice immediately that it is possible to say (in an ordinal way) that wine and cheese, for example, complement each other better than wine and toothpaste, or, beer and wine are closer substitutes than pencils and butter. These statements can be translated in the ordinal statements that the wine–cheese interaction coefficient is larger than the wine-toothpaste interaction coefficient, or that the wine–beer coefficient is smaller than the pencils–butter one.

In each of these two examples, the second pair has been deliberately chosen such as to include goods taken from groups that appear intuitively as independent (wine – toothpaste, pencils – butter). The idea is that the interaction coefficient of one pair of groups may be taken as a standard to calibrate all other coefficients. And that this pair is to be chosen among those which are independent, i.e. where there are no specific relations between the commodities from the user's point of view. By definition, the independent pairs have equal interaction coefficients.

Given a standard interaction coefficient, two groups are independent if the difference between their interaction coefficient and the standard coefficient is zero. For substitution, this difference is to be negative. For complementarity, it is positive. And since we can push the partition to the point where each group consists of only one commodity, these definitions apply also to individual commodities. The analogy with the signs of the cross-partial derivatives in the cardinal approach is perfect.

We now turn to the proof of the invariance of the difference between two coefficients of interaction.
Before proceeding, we obtain from (3.35) for $i, j \in r$

\[
\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial u}{\partial x_i} \frac{\partial^2 f_r}{\partial x_j^2} + \frac{\partial^2 u}{\partial f_r^2} \frac{\partial f_r}{\partial x_j} \frac{\partial u}{\partial x_i} + \frac{\partial^2 u}{\partial f_r^2} \left( \frac{\partial u}{\partial f_r} \right)^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}
\]

or, at equilibrium,

\[
\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial u}{\partial x_i} \frac{\partial^2 f_r}{\partial x_j^2} + \tau_{rr} \frac{\partial^2 u}{\partial f_r^2} \frac{\partial u}{\partial f_r} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}
\]

(3.38)

where

\[
\tau_{rr} = \lambda^2 \frac{\partial^2 u}{\partial f_r^2} \left( \frac{\partial u}{\partial f_r} \right)^2
\]

(3.39)

The results (3.37) and (3.39) can be presented in matrix notation in the following way. Arrange the row vector of quantities $x'$ and the row vector of prices $p'$ such that $x' = [x_1', x_2', \ldots, x_n']$ and $p' = [p_1', p_2', \ldots, p_r', p_m']$ with $x_i$ and $p_i$ being the subvectors of quantities and prices. One then defines the matrices

\[
P = \begin{bmatrix}
p_1 & 0 & \ldots & 0 & \ldots & 0 \\
0 & p_2 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \ldots & \ldots \\
0 & 0 & \ldots & 0 & \ldots & p_m \\
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
\tau_{11} & \tau_{12} & \ldots & \tau_{1m} \\
\tau_{21} & \tau_{22} & \ldots & \tau_{2m} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\tau_{m1} & \tau_{m2} & \ldots & \tau_{mm}
\end{bmatrix}
\]

where $P$ is $(n \times m)$ and $T$ is $(m \times m)$ and symmetric. The Hessian matrix of second order (direct and cross) partial derivatives can now be expressed as follows, using (3.37) and (3.39):

\[
U = U_B + PTP'
\]

(3.40)

where $U_B$ is a symmetric block diagonal matrix. The typical element of the $r$th diagonal block is the first term on the right-hand side of (3.39).
Let $v = F(u)$, $F' > 0$, to designate the whole class of differentiable utility functions associated with the same preference ordering as $u$. As in Section 1.3, the monotonic increasing transformation $F$ is assumed to be at least twice differentiable. We know that

$$
\frac{\partial^2 v}{\partial x_i \partial x_j} = F' \frac{\partial^2 u}{\partial x_i \partial x_j} + f_u \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}
$$

or, in matrix notation,

$$
V = F' U + F'' u_x u_x'
$$

where $V$ is the matrix of second order partial derivatives of $v$ and $u_x$ the vector of first order partial derivatives of $u$. At equilibrium

$$
V = F' U + F'' \lambda^2 pp'
$$

or, using (3.40)

$$
V = F' U_p + F' PT P + F'' \lambda^2 pp'
$$

$$
= F' U_p + P [F'T + F'' \lambda^2 u''] P'
$$

$$
= F' U_p + F T^a P
$$

(3.41)

where $\tau$ is a vector with all $(n)$ elements equal to unity.

In $T^a$ we have the element

$$
\tau_{rq} = F' \tau_{rq} + F'' \lambda^2
$$

(3.42)

so that, for $(s, t) \neq (r, q)$

$$
\tau_{rq}^s - \tau_{rq}^t = F'(\tau_{rq}^s - \tau_{rq}^t).
$$

(3.43)

We are glad to discover that, $F'$ being positive, the sign of the difference between two reaction coefficients remains unaltered after an increasing monotonic transformation of the utility function. Q.E.D.

It remains for us to derive the implications of separability in terms of the substitution effects, and to replace the Hicssian definitions by other (ordinal) definitions that are not contradictory with those in terms of interaction coefficients.

Remembering that the matrix of substitution effects can be written as

$$
K = \lambda U^{-1} - \left( \lambda \frac{\partial^2}{\partial y} \right) x_0 x_i'
$$
where \( x' \) is the vector of income derivatives and using (3.40), Barten (1971, Appendix A) obtains

\[
K = \lambda U_d^{-1} + X_\gamma \Phi X_\gamma
\]  
(3.44)

where

\[
\Phi = -\lambda \left[ \hat{y}^{-1}(T - TP'U^{-1}PT)\hat{y}^{-1} + \left(1/\partial y\right)\mu' \right],
\]

\[
X_\gamma = \begin{bmatrix}
\frac{\partial x_1}{\partial y} & 0 & 0 \\
0 & \frac{\partial x_r}{\partial y} & \cdots & 0 \\
0 & \cdots & \cdots \\
0 & 0 & \frac{\partial x_m}{\partial y}
\end{bmatrix}
\]

and \( \hat{y} \) is the representation of the vector \( \gamma = \lambda \mu - TP'x \) as a diagonal matrix. For \( i \in r, j \in q \) and \( r \neq q \), (3.44) implies

\[
k_{ij} = \phi_{rq} \frac{\partial x_i}{\partial y} \frac{\partial x_j}{\partial y}.
\]  
(3.45)

The factor of proportionality \( \phi_{rq} \) is the same for all cross-substitution effects between pairs of commodities with one member belonging to group \( r \) and one member belonging to group \( q \).

We know that \( k_{ij} \) is invariant under monotonic increasing transformations of the utility function. It is important to realize that here \( \phi_{rq} \) is also invariant. Indeed, replacing \( U \) by \( V \) one finds

\[
K = \left(\frac{\lambda*}{F'}\right) U_d^{-1} + X_\gamma \Phi^* X_\gamma
= \lambda U_d^{-1} + X_\gamma \Phi^* X_\gamma
\]
since

\[
\lambda* = F' \lambda, \text{ so that } \Phi = \Phi^*.
\]
The structure of preferences

When all elements of the matrix $T$ are equal, all elements of $\Phi$ are also equal. Indeed, let $T = uu'$. Then, since $i^t P x_y = p' x_y = 1$:

$$y = \lambda x - TP' x_y = \lambda x - uu' P x_y$$
$$= (\lambda y - \alpha) t = \beta t$$

with $\beta = \lambda y - \alpha$. Therefore, $\hat{\gamma} = \beta I$ and

$$\Phi = -\lambda[(1/\beta)^2 (uu' - \alpha^2 uu' P' U^{-1} Pu') + (1/\lambda) uu']$$
$$= [-\lambda(\alpha - \alpha^2 uu' P' U^{-1} p)/\beta^2 - \lambda/\lambda] uu'$$
$$= [\lambda/(\alpha - \lambda)] uu'$$

since, as follows from (2.22), $p^t U^{-1} p = 1/\lambda y$. In other words, when all groups are independent,

$$k_{ij} = \frac{\partial x_i}{\partial y} \frac{\partial x_j}{\partial y}$$  \hspace{1cm} (3.46)$$

for all $i \in r, j \in q$ and all $r$ and $q, r \neq q$. Under groupwise independence, there is one single factor of proportionality, common to all groups.

Again, we may reduce each group to one single commodity and apply (3.46) to the case of additivity, i.e. independence between individual commodities.

Exercise

3.15. (a) Check the invariance of $\phi = \lambda/(\alpha - \lambda)$ under monotonic increasing transformations of the utility function, using (3.42) and the answer to exercise 1.16d.

(b) Explain why the invariance of $\phi$ is not in contradiction to equation (3.14), where the expression $(-\lambda/(\partial \lambda/\partial y))$ is not invariant.

Finally, Barten (1971) shows that $(\phi_{rt} - \phi_{st})$ has the opposite sign of $(\tau_{rt} - \tau_{st})$, when the interaction between groups $s$ and $t$ is the standard case. If then we apply the Hicksian definitions to the difference $(\phi_{rt} - \phi_{st})$ instead of the sign of $k_{ij}$, a pair of commodities or commodity groups that are substitutes according to the cardinal definitions (and their ordinal counterpart in terms of differences between reaction coefficients) are also substitutes according to the modified Hicksian definition. Consistency in thus achieved. The reader is referred to Barten's paper for the proof and further details and qualifications.
3.5. Homothetic utility functions

To make this exposition reasonably complete, we should say a few words about homothetic utility functions, the more as we have used functions with this property in several exercises. The basic reference\(^9\) is Lau (1970) to which the reader is referred for more details and proofs.

We first define homotheticity: a function is homothetic if it can be written in the form \( u = F[F(x_1, \ldots, x_n)] \), where \( F \) is a positive, finite, continuous and strictly monotonically increasing function of one variable with \( F(0) = 0 \), and \( f \) is a homogeneous function of \( n \) variables. \( F \) is said to be positively homothetic or negatively homothetic depending on whether \( f \) is positively homogeneous or negatively homogeneous, respectively.

We immediately infer that all homogeneous utility functions are homothetic, as the function \( F \) that appears in the definition belongs to the class of admissible transformations.

**EXERCISE**

3.16. Do you know any utility function that is homothetic?

*Answer:* All utility functions in exercise 1.7 are homothetic (see exercise 1.13b). Functions (1), (2), (4), (5) and (6) in that exercise all belong to the same class. The homothetic function (3), however, belongs to another class and describes therefore another preference ordering.

3.17. Do you know any utility function that is both homothetic and additive?

*Answer:* The functions (1), (2), (4), (5) and (6) in exercise 1.7, as they are transformations of each other. The transformation (6) shows that they represent an additive preference ordering.

From the exercises based on the utility function \( u = \prod (x_i) \) and transformations of it, the reader is already familiar with the properties of homothetic utility functions. The time has come to take a closer look at these properties.

To begin with, \( \lambda' = ru \) if, and only if, the direct utility function is homogeneous of degree \( r \), where \( \lambda' = \lambda y \) and \( \lambda \) is the Lagrangian

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\(^9\) See also Samuelson (1942).
Homothetic utility functions

multiplier associated with \( u \). The proof is very simple. Write the first order conditions as:

\[
\frac{\partial u}{\partial x_i} - \lambda \frac{p_i}{y} = 0
\]

\[\sum_{i} \frac{p_i}{y} x_i - 1 = 0\]

Therefore

\[
\frac{p_i}{y} = \frac{\partial u/\partial x_i}{\lambda'}
\]

and

\[
\lambda' = \sum_i \frac{\partial u}{\partial x_i} x_i
\]

\[= ru\]  \hspace{1cm} (3.47)

by Euler's theorem. The converse is proved similarly. Lau (1970) also shows that a utility function is homothetic if, and only if, \( \lambda' = g(u) \) where \( g \) is a function of one variable.

**Exercise**

3.18. Check that (3.47) is satisfied by the utility functions of exercise 1.7.

*Hint:* To save time, use your answers to exercise 1.16a.

Homotheticity also has an important implication for the form of the demand equations: the demand equations \( x_i = \phi(p_i/y, \ldots, p_n/y) \) are *homogeneous of degree \(-1\) in the \( p_i \)'s* if, and only if, the utility function is homothetic. This implies that the demand equations can be written as:

\[
x_i = \phi_i(p_i, \ldots, p_n) y
\]

\[\hspace{1cm} (3.48)\]

where \( \phi_i \) is homogeneous of degree \(-1\) in the prices, while all commodities have constant income elasticity of one. The reader has already checked this for the class to which \( u = \prod (x_i) \) belongs in exercise 2.2.

In view of restriction (3.17), we may infer that simultaneous direct and indirect additivity implies that the utility function is homothetic. However, the converse is not true: homotheticity does not imply simultaneous direct and indirect additivity, as was emphasized by Samuelson (1965).
As noted earlier, the assumption of unitary income elasticities is entirely unrealistic: when all income elasticities are equal to one, a constant proportion of total expenditures is allocated to each commodity, which is in contradiction with facts. That is why economists are anxious to avoid homothetic utility functions in the construction of allocation models.\textsuperscript{10} The early introduction of a class of homothetic functions in this book is entirely for pedagogical reasons: these functions are easy to handle and help the student to understand the advantages of the generalization of $u = \sum \log x_i$ into the Stone-Geary function $u = \sum \beta_i \log (x_i - \gamma_i)$.

We end this discussion by drawing the reader's attention to a well-known group of utility functions, which Samuelson (1965) called the 'Bergson family' in honor of Bergson (1936) who seems to have been the first to explore their properties. These are the functions which are additive and homothetic. Here they are:

\begin{align}
    u &= \sum \beta_i \log x_i \quad \beta_i > 0, \quad \sum \beta_i = 1, \quad (3.49) \\
    u &= -\sum \beta_i x_i^\alpha \quad \beta_i > 0, \quad \alpha < 0, \quad (3.50) \\
    u &= \sum \beta_i x_i^\alpha \quad \beta_i > 0, \quad 0 < \alpha < 1. \quad (3.51)
\end{align}

Equation (3.49) obviously corresponds to $\sum \log x_i$ except for the normalization rule $\sum \beta = 1$, and is of the 'Cobb-Douglas' type.\textsuperscript{11}

EXERCISE

3.19. Let the utility function be $u = x_1^{1/2} + x_2^{1/2}$. Show that it implies expenditure proportionality.

Equation (3.49) is also implied in the 'Rotterdam model' presented in Section 2.5 (see Theil, 1967, Chapter 7 and Barten, 1967, 1969). The demonstration is due to Goldberger (1969). Goldberger first proves that if the consumer's utility function is directly additive, and if the expend-

\textsuperscript{10} These functions may be useful, though, in the derivation of a consumption function in an intertemporal context. See, e.g. Friedman (1957).

\textsuperscript{11} For further details, see Pollak (1971b). The indifference maps of (3.49), (3.50) and (3.51) are identical with the isocost maps of the constant-elasticity-of-substitution production functions and can be found in Chipman (1965).
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iterate functions \( q_i = p_i x_i = q_i(p_1, \ldots, p_m, y) \) display constant marginal budget shares (or marginal propensities to consume), so that the
\[
\mu_i = \frac{\partial q_i}{\partial y} = \frac{\partial p_i x_i}{\partial y} = \frac{p_i \partial x_i}{\partial y}
\]
are constant with respect to variation in \( y, p_1, \ldots, p_m \) the utility function must be of the Stone-Geary form, that is
\( u = \sum_i \beta_i \log (x_i - \gamma_i) \).

From exercise 1.14, we know that the associated marginal utility of income is
\[
\lambda = \frac{\sum \beta_i}{y - \sum_j p_j y_j}
\]
so that the 'income flexibility' (reciprocal of the elasticity of \( \lambda \) with respect to \( y \)) is
\[
\frac{\partial \lambda}{\partial y} = \frac{(y - \sum_j p_j y_j)}{\lambda y}.
\]

Suppose now that, as in the Rotterdam model, utility is directly additive, marginal budget shares are constant and income flexibility is constant, (with the result that \( k_{ij} \) and therefore \( S_{ij} \) is constant). We see that this requires
\[
\gamma_1 = \ldots = \gamma_i = \ldots = \gamma_n = 0,
\]
in which case the Stone-Geary function specializes to \( \sum_i \beta_i \log x_i \). The income flexibility is then equal to \(-1\).

Exercise
3.20. It should be clear to you that, for \( u = \sum_i \beta_i \log x_i \), the expenditure functions are of the form \( q_i = (\beta_i / \sum \beta_i)y \), or \( q_i = \beta_i y \) if the \( \beta \)'s are normalized.

Hint: Look at the answers to exercises 1.14 and 1.11.

In view of the lack of realism in the Bergson functions, one may question the acceptability of the Rotterdam model. Some authors,

\[\text{12} \text{ The reader should notice that, in empirical applications, the Rotterdam model is not written in the continuous version given in equation (2.43). In fact, a finite approximation is used. That is the reason why the estimated elasticities are not unitary.}\]
such as Yoshihara (1969) simply reject it. Barten (1969, p. 13, 14) argues that it remains a useful tool for testing the general restrictions. He thinks that the model can be justified as the first terms in a Taylor expansion of arbitrary demand functions and that the approximation will be good enough provided real income and relative prices do not change too much over the period of estimation. Barten also emphasizes that, in applied work, one is always using data of an aggregated type and that it is therefore of limited interest to look for the underlying utility function. If this were true, the efforts made in this chapter (and throughout this book) to derive particular restrictions from specified utility functions would lose most of their empirical relevance. This raises the problem of aggregation over individuals, about which more will be said in the next chapter.