7.1. Introduction

The short-run theory of cost with fixed input prices is perhaps one of the best understood aspects of economic theory. It is correctly, and usually exhaustively, explored in even the most elementary texts. I have nothing new to add to the conventional accounts.\(^1\) For that reason, the short-run theory of cost is entirely ignored.

The long-run theory of cost with fixed input prices is almost as well established as the short-run theory. A graphical analysis of the theory is accordingly omitted, and the mathematical analysis is as restricted as possible.

7.1.1 The expansion path and long-run cost

Assume that the firm under consideration produces a single output under conditions given by the following production function:

\[ q = f(x_1, x_2, \ldots, x_n), \quad (7.1.1) \]

where the \(x_i\)'s represent the quantities of inputs \(X_i\). The assumptions concerning the production function, which were introduced in chapter 4, are retained throughout. Furthermore, by assumption the firm is a perfect competitor in each input market. Hence the prices of the inputs are given and fixed, i.e.

\[ p_i = \bar{p}_i \quad (i = 1, 2, \ldots, n), \quad (7.1.2) \]

where \(\bar{p}_i\) is a constant.

In the 'long run'\(^2\) all inputs are variable. Hence the least cost of producing each level of output is achieved when factor proportions are adjusted so that

\[ \frac{f_1}{p_1} = \frac{f_2}{p_2} = \ldots = \frac{f_n}{p_n}. \quad (7.1.3) \]

\(^1\) For example, see Ferguson [1966a].

\(^2\) For the most part, the qualifying term 'long run' is omitted in the ensuing discussion. The reader should remember that all references to cost, unless otherwise explicitly stated, refer to long-run cost.

\(^3\) See chapter 6.
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These equations establish the expansion path in input space, a locus such as OABD in figure 39. Mapping the expansion path from input space into cost-output space generates the total cost curve. For example, if $r$ is the price of a unit of capital service, one point is given by $Q_0$, $[r(OK_0) + w(OL_0)]$; another is given by $Q_1$, $[r(OK_0) + w(OL_0)]$, etc. Thus the long-run total cost curve is merely the output-space equivalent of the expansion path. The corresponding average and marginal cost curves are obtained in the usual way.

7.1.2 Viner revisited

Since the seminal paper by Viner, the long-run average cost curve has been portrayed as the envelope of short-run average cost curves. It is indeed the envelope; but explaining why it is sometimes becomes tedious. Yet it may be done quite simply in the two-input case (see figures 39 and 40).

\footnote{Viner [1931].}
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In figure 39 the $Q_i$'s are isoquants representing successively larger volumes of output, and the $P_i P_i'$'s are isocost curves representing successively larger expenditures at constant input prices. The tangency conditions, equations (7.1.3), generate the expansion path $OABD$, which may be mapped into output space as the total cost curve. $LAC$ in figure 40 is the corresponding long-run average cost curve.

Now suppose for the moment that we are in a ‘short-run’ situation in which the quantity of capital is fixed at $K_e K_e'$. If production occurs at point $B$, the short-run and long-run costs are identical. However, any short-run variation of output along $K_e K_e'$ will give rise to higher costs than are entailed when input proportions are optimally adjusted (i.e., equations (7.1.3) hold only at point $B$ on $K_e K_e'$). For example, if the $Q_e$-level of output is produced, the firm must operate at $A'$ rather than $A$. Total cost is represented by $C_e C_e' > P_i P_i'$; hence short-run average cost exceeds long-run, or optimal adjustment, average cost. A similar statement applies to production at $D'$ on $Q_{e'}$, with short-run cost represented by $C_{e'} C_{e'}'$.

Since the isoquants $Q_e$ and $Q_{e'}$ are arbitrary, the above analysis applies to all levels of output above and below $Q_e$. Hence the short-run average cost curve lies above the long-run average cost curve at every point other than $B$, and it is precisely tangent at $B$. By varying the ‘size of plant’, i.e., $K_e K_e'$, the ‘envelope theorem’ is established.

Fig. 40

point $B$, the short-run and long-run costs are identical. However, any short-run variation of output along $K_e K_e'$ will give rise to higher costs than are entailed when input proportions are optimally adjusted (i.e., equations (7.1.3) hold only at point $B$ on $K_e K_e'$). For example, if the $Q_e$-level of output is produced, the firm must operate at $A'$ rather than $A$. Total cost is represented by $C_e C_e' > P_i P_i'$; hence short-run average cost exceeds long-run, or optimal adjustment, average cost. A similar statement applies to production at $D'$ on $Q_{e'}$, with short-run cost represented by $C_{e'} C_{e'}'$.

Since the isoquants $Q_e$ and $Q_{e'}$ are arbitrary, the above analysis applies to all levels of output above and below $Q_e$. Hence the short-run average cost curve lies above the long-run average cost curve at every point other than $B$, and it is precisely tangent at $B$. By varying the ‘size of plant’, i.e., $K_e K_e'$, the ‘envelope theorem’ is established.
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7.2 Analysis of long-run cost

Equations (7.1.3) and (7.1.1) together provide \( n \) equations in the \( n \) input quantities \( x_1, x_2, \ldots, x_n \) and output \( q \), given the input prices. Hence these equations implicitly define input quantities as functions of output along the expansion path:

\[
x_i = x_i(q) \quad (i=1, 2, \ldots, n).
\]

(7.2.1)

Thus the production function may be written

\[
q = f[x_1(q), x_2(q), \ldots, x_n(q)].
\]

(7.2.2)

Differentiating equation (7.2.2) with respect to \( q \) yields

\[
\sum_{i=1}^{n} \frac{f_i}{dq} = 1.
\]

(7.2.3)

7.2.1 Cost functions

The total cost of production is given by

\[
c = \sum_{i=1}^{n} p_i x_i(q) = c(q).
\]

(7.2.4)

Suppose from some initial point on the expansion path, the inputs are given the arbitrary increments \( dx_1, dx_2, \ldots, dx_n \). From equation (7.2.4) cost expands (or contracts) by

\[
dc = \sum_{i=1}^{n} p_i dx_i.
\]

(7.2.5)

From equations (7.1.1), output expands (or contracts) by

\[
dq = \sum_{i=1}^{n} f_i dx_i.
\]

(7.2.6)

At this point it is convenient to introduce the following

Definition: Marginal cost is the first derivative of the total cost function with respect to output.

From equations (7.2.5) and (7.2.6), marginal cost may always be expressed as

\[
\frac{dc}{dq} = \frac{\sum_{i=1}^{n} p_i dx_i}{\sum_{i=1}^{n} f_i dx_i}.
\]

(7.2.7)

However, along the expansion path the expression for marginal cost is simpler. In full, the expansion path is defined by

\[
\frac{f_i}{p_i} = \frac{1}{\lambda} \quad (i=1, 2, \ldots, n),
\]

(7.2.8)

1 Throughout sections 7.2 and 7.3 all equations hold only along the expansion path (or long-run cost curve). Hence this qualifying term is usually omitted.
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where $\lambda$ is the Lagrange multiplier in the constrained cost minimization problem. Thus $p_i = \lambda f_i$ may be substituted in equation (7.2.5), obtaining

$$dc = \sum_{i=1}^{n} p_i dx_i = \lambda \sum_{i=1}^{n} f_i dx_i.$$  \hspace{1cm} (7.2.9)

Thus the ratio in equation (7.2.7) reduces to

$$\frac{dc}{dq} = c' = \lambda,$$  \hspace{1cm} (7.2.10)

a result previously established in chapter 4.

The same result may be obtained by taking the derivative of equation (7.2.4) straightforwardly:

$$\frac{dc}{dq} = c' = \sum_{i=1}^{n} p_i \frac{dx_i}{dq}.$$  \hspace{1cm} (7.2.11)

Substituting $\lambda f_i = p_i$ in equation (7.2.11) yields

$$c' = \lambda \sum_{i=1}^{n} f_i \frac{dx_i}{dq}.$$  \hspace{1cm} (7.2.12)

Substituting equation (7.2.3) in equation (7.2.12) immediately yields equation (7.2.10).

Further information about marginal cost may be obtained by substituting equation (7.2.8) in equation (7.2.10), obtaining

$$c' = \frac{p_i}{f_i} (i = 1, 2, \ldots, n).$$  \hspace{1cm} (7.2.13)

The result in equation (7.2.13) may be expressed as the following

Relation: Along the expansion path, marginal cost is the ratio of an input's price to its marginal product. Thus on the expansion path, marginal cost is the same irrespective of the input for which it is calculated.

At present there is less to say about average cost, which is merely introduced by the following

Definition: Average cost is the cost per unit of output, or the ratio of total cost to total output. Thus average cost is given by

$$\frac{c}{q} = \frac{c(q)}{q} = \bar{c}(q).$$  \hspace{1cm} (7.2.14)

7.2.2 Elasticities of the cost functions

Let $\kappa$ denote the elasticity of the total cost function. Thus by definition:

$$\kappa = \frac{dc}{dq} \cdot \frac{1}{c} = \frac{c'}{c}.$$  \hspace{1cm} (7.2.15)

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As customary with elasticities, the elasticity of total cost is the ratio of marginal cost to average cost.¹ ²

Next, the elasticity of average cost is given by

\[
\gamma = \frac{dc}{dq} \bar{c} = \frac{c'}{\bar{c}} - 1 = \kappa - 1. \tag{7.2.16}
\]

Thus the elasticity of average cost is equal to the elasticity of total cost minus one. This relation provides an interesting contrast between elasticities of demand and of cost. The elasticity of demand refers to the average revenue curve. The value of the elasticity of average revenue may be used, in comparison with unity, to infer the behavior of total revenue. The elasticity of cost has the opposite application. Specifically, the value of the elasticity of cost may be used, in comparison with unity, to infer the behavior of the average cost function (and thus the nature of returns to scale along the expansion path).

Finally, we may determine the elasticity of marginal cost. From equation (7.2.15),

\[
c' = \bar{c}\kappa. \tag{7.2.17}
\]

From the conventional 'rule of elasticities', ³

\[
\text{El. } c' = \text{El. } \bar{c} + \text{El. } \kappa. \tag{7.2.18}
\]

Setting El. \(c'\equiv \rho\) and El. \(\kappa = \omega\), equation (7.2.18) may be written as

\[
\rho = \kappa - 1 + \omega. \tag{7.2.19}
\]

7.2.3 Cost elasticities and the function coefficient

The cost elasticities may be expressed directly in terms of the function coefficient (c); to do so shows the importance of the reciprocal of the elasticity of cost.⁴ Use equations (7.2.8) in equation (7.2.4) to write the cost function as

\[
c = \sum_{i=1}^{n} p_i x_i = \lambda \Sigma f_i x_i. \tag{7.2.20}
\]

From chapter 4 we have the following definition of the function coefficient:

\[
c = \frac{\Sigma f_i x_i}{q}, \tag{7.2.21}
\]

¹ The elasticity can also be derived more precisely from equation (7.2.4):

\[
\frac{dc}{dq} = \frac{q}{c} \left[ \sum_{i=1}^{n} \frac{p_i}{q} dx_i \right] = \frac{q}{c} \left[ \lambda \Sigma f_i dx_i \right] = \lambda \frac{q}{c} = \frac{c'}{\bar{c}}.
\]

³ Moore [1929, p. 77] was presumably the first to introduce the elasticity of cost, which he described as 'the coefficient of relative cost of production'. Moore also described the reciprocal of the elasticity of cost as 'the coefficient of relative efficiency of organization'. The reciprocal of the elasticity of cost had previously been used by Johnson [1913, p. 358] and by Bowley [1924, p. 34]. Also see Allen [1938, pp. 260-2]. On the importance of the reciprocal of the elasticity of cost, see subsection 7.2.3 below.

⁴ See footnote 2 above.
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Using equation (7.2.21) and the definition \( \lambda = c' \) in equation (7.2.20) yields

\[
e = qe'e. \tag{7.2.22}
\]

Thus total cost at any point on the expansion path is the product of quantity produced, marginal cost, and the function coefficient. Needless to say, equation (7.2.22) holds only for positive values of \( \varepsilon \) because the expansion path lies strictly within the economic region of production, i.e. where \( \varepsilon > 0 \).

A similar relation holds for average cost. Dividing both sides of equation (7.2.22) by \( q \) yields

\[
\bar{c} = c'e, \tag{7.2.23}
\]

or average cost is the product of marginal cost and the function coefficient. Since \( c' \) and \( q \) are positive, equations (7.2.22) and (7.2.23) may at first seem surprising. Indeed, on the surface these equations seem to indicate that in a given factor point, the greater the instantaneous returns to scale, the greater are total and average cost! This interpretation is not valid, of course, because \( c, \bar{c} \) and \( c' \) are themselves functions of \( \varepsilon \).

To get at the proper relation, divide both sides of equation (7.2.23) by \( c' \) and take the reciprocal, obtaining

\[
\kappa = \frac{1}{c'}. \tag{7.2.24}
\]

Equation (7.2.24) provides the correct interpretation, which is summarized by the following

Relation: The smaller is the instantaneous measure of returns to scale, the greater is the proportional increase in cost relative to a given proportional increase in output.

In a regular production function (see chapter 4), \( \varepsilon \) varies inversely with output. For small \( q \), \( \varepsilon > 1 \); as \( q \) increases, \( \varepsilon = 1 \) at a point and \( \varepsilon < 1 \) thereafter. Thus equation (7.2.24) also leads to the following

Relation: Given a regular production law, cost first increases at a decreasing rate, passes through a point of inflection, and increases at an increasing rate thereafter.

This relation is illustrated in figure 41. Next, equations (7.2.16) and (7.2.24) may be used to show the relation between the elasticity of average cost and the function coefficient:

\[
\gamma = \kappa - 1 = \frac{1}{\varepsilon} - 1 = \frac{1 - \varepsilon}{\varepsilon}. \tag{7.2.25}
\]

Finally, the relation between the elasticity of marginal cost and the function

\footnote{Thus, for example, one cannot say \( d\varepsilon/d\varepsilon = qe' \).}
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coefficient may be determined from equation (7.2.19):
\[
\rho = \kappa + 1 + \omega = \frac{1 - \varepsilon}{\varepsilon} - \theta_{\varepsilon}
\]  

(7.2.26)

where \( \theta_{\varepsilon} \) is the elasticity of the function coefficient.\(^1\)

Let us now discuss the implications of equations (7.2.25) and (7.2.26), which are illustrated in figure 42. For the moment, assume that the production function follows a regular production law. When \( \varepsilon > 1, \gamma < 0 \). Therefore, a given proportional increase in output causes a smaller proportional reduction in average cost.\(^2\) On the other hand, when \( \varepsilon < 1, \gamma > 0 \). In this case, a given proportional increase in output causes an increase in average cost that is less than proportional or greater than proportional according as \( \varepsilon \geq \frac{1}{2} \). It is obvious from equation (7.2.25) that \( \gamma = 0 \) when \( \varepsilon = 1 \). Thus there is an extreme point on the average cost curve corresponding to \( \varepsilon = 1 \); in light of the analysis just above, the extremum must be a minimum.

\(^1\) This follows from the ‘rule of elasticity’, as follows:

\[\text{El.} \varepsilon = \frac{\varepsilon - 1}{\varepsilon} - \text{El.} \varepsilon = -\theta_{\varepsilon}, \text{ or } \omega = -\theta_{\varepsilon}.\]

\(^2\) When the elasticity of average cost is negative, it must be inelastically negative, i.e. when \( \varepsilon > 1, \frac{\varepsilon - 1}{\varepsilon} < 1 \).
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Turn now to equation (7.2.26). The magnitude of $\theta_e$ cannot be determined without specifying a particular production function. However, it is known that in a general production law, $\delta e/dq < 0$. Hence $\theta_e < 0$, and $-\theta_e > 0$ over the relevant range of production. On the other hand, if the production function is homogeneous (see chapter 5), $\epsilon$ is a constant and $\theta_e = 0$.

When $\epsilon > 1$, the elasticity of marginal cost may be negative. This occurs if returns to scale are very strong, so that the negative term $(1 - \epsilon)/\epsilon$ dominates the necessarily positive term $-\theta_e$. However, it must become positive at some output rate while $\epsilon > 1$ because of the influence of $-\theta_e > 0$, and a fortiori it is positive for $\epsilon < 1$. Further, a comparison of equations (7.2.25) and (7.2.26) reveals that the marginal cost function must always be more elastic than the average cost function, given a regular production law. When $\epsilon = 1$, $\theta_e$ is instantaneously zero, marginal cost and average cost are therefore equal.

These results for a regular production law may be summarized as the following:

Relations: (i) Average cost diminishes when $\epsilon > 1$, attains its minimum when $\epsilon = 1$, and increases when $\epsilon < 1$; (ii) when the elasticity of average cost is negative, it must be inelastically negative; however it is positively elastic for $\epsilon < \frac{1}{4}$; (iii) marginal cost may have a declining range; however
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it must turn up before \( \varepsilon = 1 \); (iv) at \( \varepsilon = 1 \), marginal cost equals average cost; and (v) the marginal cost function is always more elastic than the average cost function unless the production function is homogeneous.

As shown in chapter 5, \( \varepsilon \) is a constant when the production function is homogeneous. Indeed, with a homogeneous production function, there exist increasing, constant, or decreasing returns to scale according as \( \varepsilon \geq 1 \). From equations (7.2.25) and (7.2.26), it is apparent that the elasticities of average and marginal cost are identical when \( \varepsilon \) is a constant (i.e. \( \theta_\varepsilon = 0 \)). Further, the curves are negatively or positively sloped according as \( \varepsilon \geq 1 \); they are horizontal when \( \varepsilon = 1 \).

Let us summarize these results as the following

*Relations:* (i) when the production function is homogeneous, the elasticities of marginal and average cost are the same; (ii) the functions are negatively or positively sloped according as there are increasing or decreasing returns to scale; (iii) if the production function is homogeneous of degree one, there are constant returns to scale; in this case, average and marginal cost are identical and constant. 1

7.3 Two examples of cost functions

From the discussion above it is apparent that with fixed input prices, total cost is a function of output and of nothing else. But output is determined by the production function. Thus if a specific production function is known, the associated cost function can always be derived. As will be obvious, direct derivation of cost functions can become a messy business even for very simple production functions.

7.3.1 The Cobb–Douglas function

In this subsection the cost function associated with a Cobb–Douglas production function is derived and examined in detail. In the following subsection, the CES function is treated in a cursory manner.

Write the Cobb–Douglas function in the form

\[
q = ax_1^\alpha x_2^\beta \quad (0 < \alpha, \alpha, \beta).
\]

(7.3.1)

Let \( p_1 \) and \( p_2 \) represent the constant unit prices of the inputs \( X_1 \) and \( X_2 \). From equation (7.3.1) equality between the marginal rate of technical substitution and the input-price ratio, the condition defining the expansion path, requires that

\[
\frac{P_1}{P_2} = \frac{ax_2}{\beta x_1}. \tag{7.3.2}
\]

Next, take the logarithms of equations (7.3.1) and (7.3.2) and write them as

1 On this score, see Samuelson [1947, pp. 78–80].
a pair of simultaneous equations:
\[
\begin{align*}
\alpha \ln x_1 + \beta \ln x_2 &= \ln q - \ln a, \\
-\ln x_1 + \ln x_2 &= \ln \beta - \ln \alpha + \ln p_1 - \ln p_2.
\end{align*}
\] (7.3-3)

Solving equations (7.3-3) simultaneously for \(x_1\) and \(x_2\) yields
\[
\begin{align*}
x_1^* &= (\alpha^{-1} q^{\alpha-\beta} a^{\beta} p_1^p p_2^p)^{1/(\alpha+\beta)}, \\
x_2^* &= (\alpha^{-1} q^{\beta-\alpha} a^{\alpha} p_1^p p_2^{1-p})^{1/(\alpha+\beta)},
\end{align*}
\] (7.3-4)

where \(x_1^*\) and \(x_2^*\) are the quantities of the inputs required to produce \(q\) units of output at the cost-minimizing input ratio (given by equation (7.3-2)).

The cost of producing \(q\) units of output is
\[
c = p_1 x_1^* + p_2 x_2^*.
\] (7.3-5)

Substituting equations (7.3-4) in equation (7.3-5) yields
\[
c = p_2 \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)} + p_1 \left[ \left( \frac{\beta p_2}{\alpha p_1} \right)^{\frac{\beta}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)}.
\] (7.3-6)

Now consider the following reduction of the rightmost term in equation (7.3-6):
\[
\begin{align*}
p_1 \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)} &= \frac{p_1 \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)}}{\left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\beta}{\alpha+\beta}} \frac{q}{a}} = \frac{p_2}{\beta} \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\beta}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)}.
\end{align*}
\] (7.3-7)

Substituting equation (7.3-7) in equation (7.3-6), one obtains
\[
c = p_2 \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)} + \frac{\alpha p_2}{\beta} \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\beta}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)},
\] (7.3-8)

or
\[
c = p_2 \left( \frac{\alpha+\beta}{\beta} \right) \left[ \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} \frac{q}{a} \right]^{1/(\alpha+\beta)}.
\] (7.3-9)

Since the technological parameters \(\alpha\), \(\alpha\), and \(\beta\) are given, as well as the market parameters \(p_1\) and \(p_2\), equation (7.3-9) shows cost as a function of output alone.

Average and marginal cost may be obtained immediately from the cost function (equation (7.3-9)):
\[
\bar{c} = \frac{\alpha+\beta}{\beta} \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} \left( \frac{1}{a} \right)^{\frac{\alpha}{\alpha+\beta}} \left( \frac{1-\alpha-\beta}{a} \right)^{\frac{\beta}{\alpha+\beta}}.
\] (7.3-10)
and
\[ c' = \frac{1}{\alpha + \beta} \left[ p_a \left( \frac{\alpha + \beta}{\beta} \right) \left( \frac{\beta p_a}{\alpha p_a} \right) \frac{\alpha}{\alpha + \beta} \left( \frac{1}{\alpha} \right) \frac{1}{q} \frac{1 - \alpha - \beta}{\alpha + \beta} \right]. \] (7.3.11)

Similarly, the elasticity of cost is
\[ \kappa = \frac{c'}{c} = \frac{1}{\alpha + \beta}, \] (7.3.12)

the exponent of \( q \) in the cost function. In like manner, the elasticity of average cost is
\[ \gamma = \kappa - 1 = \frac{1 - \alpha - \beta}{\alpha + \beta}, \] (7.3.13)

which is also the exponent of \( q \) in the average cost function. Finally, since \( \theta = 0 \) for homogeneous functions (i.e., the function coefficient is a constant), the elasticity of marginal cost is
\[ \rho = \gamma = \kappa - 1 = \frac{1 - \alpha - \beta}{\alpha + \beta}. \] (7.3.14)

One readily sees from equations (7.3.12)-(7.3.14) that the degree of homogeneity is of the utmost importance. This fact may also be seen directly from the average and marginal cost functions and their derivatives.

For convenience, let
\[ b = p_a \left( \frac{\alpha + \beta}{\beta} \right) \left( \frac{\beta p_a}{\alpha p_a} \right) \frac{\alpha}{\alpha + \beta} > 0. \] (7.3.15)

Then
\[ c'' = \left[ \frac{1 - (\alpha + \beta)}{\alpha + \beta} \right] b q^{\frac{1 - 2(\alpha + \beta)}{\alpha + \beta}}, \] (7.3.16)

\[ c' = \left[ \frac{1 - (\alpha + \beta)}{\alpha + \beta} \right] b q^{\frac{1 - 2(\alpha + \beta)}{\alpha + \beta}}, \] (7.3.17)

\[ c'' = \left( \frac{1 - 3(\alpha + \beta) + 2(\alpha + \beta)^2}{(\alpha + \beta)^2} \right) b q^{\frac{1 - 2(\alpha + \beta)}{\alpha + \beta}}, \] (7.3.18)

and
\[ c'' = \left( \frac{1 - 3(\alpha + \beta) + 2(\alpha + \beta)^2}{(\alpha + \beta)^2} \right) b q^{\frac{1 - 2(\alpha + \beta)}{\alpha + \beta}}. \] (7.3.19)

Further, let us note that
\[
1 - 3(\alpha + \beta) + 2(\alpha + \beta)^2 \begin{cases} > 0 & \text{for } (\alpha + \beta) > 1.0, \\ < 0 & \text{for } 0.5 < (\alpha + \beta) < 1.0, \\ > 0 & \text{for } (\alpha + \beta) < 0.5. \end{cases}
\] (7.3.20)

The properties of the average and marginal cost curves may now be determined from equations (7.3.10)-(7.3.20). First, from equations (7.3.10) and (7.3.11), marginal cost is greater than or less than average cost
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According as there are decreasing or increasing returns to scale (i.e. according as \((\alpha + \beta) \leq 1\)). Next, from equations (7.3.16) and (7.3.17), the marginal and average cost curves rise or fall according as there are decreasing or increasing returns to scale. Finally, consider equations (7.3.18)–(7.3.19) and inequality (7.3.20). If increasing returns to scale prevail, both the average and marginal cost curves decline and are concave from above. If there are decreasing returns to scale such that

\[ 0.5 < (\alpha + \beta) < 1, \]

both curves rise and are concave from below. On the other hand, if there are strongly decreasing returns to scale, i.e. \((\alpha + \beta) < 0.5\), the curves not only rise but are concave from above.

The very special case of constant returns to scale has not yet been mentioned. If \((\alpha + \beta) = 1\), equation (7.3.9) reduces to

\[ c = bq. \]  

(7.3.21)

Thus cost is a linear function of output, as would be expected. Average and marginal costs are constant and equal, as shown by

\[ c' = \bar{c} = b, \]  

(7.3.22)

and

\[ \kappa = 1, \quad \gamma = \rho = 0. \]  

(7.3.23)

7.3.2 The CES function

The linearly homogeneous form of the CES function may be written

\[ q = \gamma \left[ \delta x_1^{\beta} + (1 - \delta) x_2^{\beta} \right]^{-1/\rho}. \]  

(7.3.24)

The cost function, in this case, is somewhat more difficult to derive. The expansion path is defined by

\[ \left( \frac{x_1}{x_2} \right)^{-1+\rho} = \left( \frac{1 - \delta}{\delta} \right) \left( \frac{p_1}{p_2} \right)^{\rho}. \]  

(7.3.25)

Thus, using the notation above,

\[ x_1^* = \left( \frac{1 - \delta}{\delta} \right)^{-\rho} \left( \frac{p_1}{p_2} \right)^{-\rho} x_2^*. \]  

(7.3.26)

Substitute equation (7.3.26) in equation (7.3.24) to obtain

\[ q = \gamma \left[ \delta \left( \frac{1 - \delta}{\delta} \right)^{-\rho} \left( \frac{p_1}{p_2} \right)^{-\rho} x_2^* \right]^{-1/\rho}. \]  

(7.3.27)
7.3 EXAMPLES

Solving yields
\[ x^*_2 = \frac{q}{\gamma} \left[ \delta \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} \left( \frac{p_1}{p_2} \right)^{1-\sigma} + (1-\delta) \right]^{\sigma/(1-\sigma)} \]  \hspace{1cm} (7.3.28)

Now substitute equation (7.3.28) in equation (7.3.26), obtaining
\[ x^*_1 = \frac{q}{\gamma} \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} \left( \frac{p_1}{p_2} \right)^{1-\sigma} \left[ \delta \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} \left( \frac{p_1}{p_2} \right)^{1-\sigma} + (1-\delta) \right]^{\sigma/(1-\sigma)} \]  \hspace{1cm} (7.3.29)

Since
\[ c = p_1 x^*_1 + p_2 x^*_2, \]  \hspace{1cm} (7.3.30)

it follows that
\[ c = \frac{q}{\gamma} \left[ \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} p_1^{1-\sigma} p_2^{1-\sigma} + p_2 \right] \left[ \delta \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} \left( \frac{p_1}{p_2} \right)^{1-\sigma} + (1-\delta) \right]^{\sigma/(1-\sigma)} \]  \hspace{1cm} (7.3.31)

The first term in brackets on the right-hand side of equation (7.3.31) may be written
\[ \left[ \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} p_1^{1-\sigma} p_2^{1-\sigma} + p_2 \right] = p_1 \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} \left( \frac{p_1}{p_2} \right)^{1-\sigma} + p_2 \]  \hspace{1cm} (7.3.32)

After some considerable manipulation, the second term in brackets on the right-hand side of equation (7.3.31) may be written
\[ \left[ \delta \left( \frac{1-\delta}{\delta} \right)^{1-\sigma} \left( \frac{p_1}{p_2} \right)^{1-\sigma} + (1-\delta) \right]^{\sigma/(1-\sigma)} \]  \hspace{1cm} (7.3.33)

Finally, substituting equations (7.3.32) and (7.3.33) in equation (7.3.31) yields the CES cost function:
\[ c = \left( \frac{1-\delta}{p_2} \right)^{\sigma/(1-\sigma)} \left[ p_1 \left( \frac{\delta}{1-\delta} \right)^{1-\sigma} \left( \frac{p_2}{p_1} \right)^{1-\sigma} + p_2 \right]^{\sigma/(1-\sigma)} \frac{q}{\gamma}. \]  \hspace{1cm} (7.3.34)

As in the linearly homogeneous Cobb–Douglas case, cost is a linear function of output. Thus marginal and average cost are equal and the cost elasticities have the same value (given fixed input prices). The only essential difference between the Cobb–Douglas and CES cost functions lies in the appearance of the elasticity of substitution in the latter.

Since all terms in equation (7.3.34) are positive, one may easily show that
\[ \frac{dc}{d\gamma} < 0, \]  \hspace{1cm} (7.3.35)

and
\[ \frac{dc}{d\sigma} < 0. \]  \hspace{1cm} (7.3.36)
THEORY OF COST

Inequality (7.3.35) implies that cost varies inversely with the 'efficiency' parameter, as should be expected. Inequality (7.3.36) shows that, all other things equal, the greater the elasticity of substitution, the lower the associated level of cost.\footnote{1}

The result $dcd/e < 0$ for static cost functions is closely related to the result that the rate of growth of output, given technical progress, is positively associated with the value of $c$. See Ferguson [1965d].

The result $dcd/e < 0$ is logical and virtually compelling. That is, the greater the efficiency of production, the lower the associated level of total cost. In the first draft of this chapter, I used the CES given by Walters [1963]. Professor John Moroney discovered that his function implied $dcd/e > 0$ and insisted that my equation be changed. I am much indebted to him on this point. Direct derivation yielded the equation in the text. Further, limit analysis shows that the CES function reduces to the Cobb-Douglas cost function as $c \to 1$. In Walters' function, as $c \to 1, e \to q$. Thus the function in Walters' paper must be misprinted.

\footnote{1}{The result $dcd/e < 0$ for static cost functions is closely related to the result that the rate of growth of output, given technical progress, is positively associated with the value of $c$. See Ferguson [1965d].}