THE SIMPLE MATHEMATICS OF LINEAR PRODUCTION MODELS

By

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I. The Simple Mathematics of Linear Production Models

4. Vectors

1. A vector is an ordered set of numbers. By an ordered set we mean a set which is distinguished not only by the elements it contains, but also by the order in which they appear. We shall denote a general n-component vector by \( \mathbf{a} = (a_1, a_2, \ldots, a_n) \) where the \( a_i \) [\( i = 1, 2, \ldots, n \)] stand for the \( n \) components.

2. Let us denote a row vector as \( \mathbf{a} = [a_1, \ldots, a_n]_{n \times 1} \).

3. Let us denote a column vector as \( \mathbf{a}' = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \).

4. \( \mathbf{a}' \) is called the transpose of \( \mathbf{a} \); that is, transpose changes a row vector into a column vector and vice versa. The latter point is denoted in the following manner: \( (\mathbf{a}')' = \mathbf{a} \).

2. Null vector - it is a vector in which every component is zero:

   \( \mathbf{0} = (0, \ldots, 0)_{1 \times n} \)

3. Unit vector - it is a vector in which the \( i \)th component is the number one (1) and all the other components are zeros. There are hence \( n \) unit vectors of order \( n \):

   \( \mathbf{e}_1 = (1, 0, \ldots, 0)_{1 \times n}, \ldots, \mathbf{e}_n = (0, \ldots, 0, 1)_{1 \times n} \)

4. Sum vector - it is defined as a vector in which every component is equal to unity: \( \mathbf{1} = (1, \ldots, 1)_{1 \times n} \).

5. Equality - two vectors \( \mathbf{a} \) and \( \mathbf{b} \) are equal if and only if each component of vector \( \mathbf{a} \) is equal to the corresponding component of vector \( \mathbf{b} \).
b: \ \ a_i = b_i \ \ \text{for all } i.

a. reflexive property: \ a = a
b. symmetric property: \ \text{if } a = b, \ \text{then } b = a
c. transitive property: \ \text{if } a = b \ \text{and } b = c, \ \text{then } a = c

d. scalar multiplication: given a scalar \ \lambda \ \text{and a vector } a, \ \text{their product is defined as another vector } \ \lambda a, \ \text{the components of which are the components of vector } a \ \text{multiplied by the scalar } \ \lambda:

\lambda a = (\lambda a_1, \ldots, \lambda a_n)

e. addition: given two \ n\text{-component vectors } a \ \text{and } b, \ \text{their sum is defined as the vector } (a+b), \ \text{the components of which are the sum of the corresponding components of the two vectors being added:}

a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)

f. commutative property: \ a + b = b + a

g. associative property: \ (a + b) + c = a + (b + c)
h. distributive property: \ \lambda(a + b) = \lambda a + \lambda b

(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a

8. Vector space

a. To define this concept and bring out its importance we first need to define the following terms:

i. linear combination: (i) vector \ a \ can be expressed as a linear combination of other vectors, i.e. a combination obtained by applying the operations of addition and scalar multiplication:

\ a = (a_1, \ldots, a_n) = a_1(1, \ldots, 0) + \ldots + a_n(0, \ldots, 0, 1); \ (ii) \ \text{more generally, there exists a } n\text{-component vector } z \ \text{whose } \ n\text{-components are } n\text{-components vectors.} \ \text{Hence we say that vector } z \ \text{is a linear combination of the } a_j \ \text{s.} \}
is a linear combination of the vectors \( a_1, \ldots, a_n \).

\[ z = \lambda_1 a_1 + \cdots + \lambda_m a_m \]

where \( a_i = (a_{i1}, \ldots, a_{in}) \).

2. Linearly dependence - consider the following special linear combination: \( 0 = \lambda_1 a_1 + \cdots + \lambda_m a_m \). If the equation can be satisfied with at least one \( \lambda_i \neq 0 \), then the \( m \) vectors are linearly dependent. If the vectors \( a_1, \ldots, a_m \) are linearly dependent, then at least one of them can be expressed as a linear combination of the others. For example, suppose that \( \lambda_i \neq 0 \), we can write

\[ a_i = \frac{\lambda_2}{\lambda_1} a_2 + \cdots + \frac{\lambda_m}{\lambda_1} a_m \]

3. Linearly independence - if \( 0 = \lambda_1 a_1 + \cdots + \lambda_m a_m \) can occur only if \( \lambda_1 = \cdots = \lambda_m = 0 \), then none of the vectors \( a_1, \ldots, a_m \) can be expressed as a linear combination of the other.

A vector space is a set of \( z \) vectors which is obtained from the vectors \( a_1, \ldots, a_n \) by varying \( \lambda_1, \ldots, \lambda_m \). The set of vectors satisfy the following conditions:

(i) the operation of addition is defined for any two components of the set;

(ii) the operation of multiplication by a scalar is defined for any component of the set and any scalar; and

(iii) the set is closed under these operations.

Given a vector space there exists any number of sets of vectors which span it, i.e., every vector of the vector space can be written as a linear combination of the vectors in the particular set.
(2) It is of interest to select from all sets which span the vector space those sets which contain the least number of vectors. Thus it will be necessary to eliminate all linearly dependent vectors in order to be left with linearly independent vectors. Consequently all that will be left will be those sets of vectors consisting entirely of linearly independent vectors and which span the vector space. Each one of these specific vectors is called a basis of the vector space.

(3) The dimension of a vector space—denoted as $V^n$—is the maximum number of linearly independent vectors which can be found in that space. Thus any linearly independent set of $n$ vectors which span the space will also be a basis for that space.

4. Vector multiplication

a. Let $V^n$ be an $n$-dimensional vector space and $a$ and $b$ be two $n$-component row vectors, then the scalar product can be defined as the multiplication of the two vectors:

$$a \cdot b = (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^{n} a_i b_i$$

(1) If $a \cdot b = 0$, then they are said to be orthogonal.
(2) If $a \cdot a = 1$, then they are said to be orthonormal.

b. Scalar product properties

(1) $(a + b) \cdot c = a \cdot c + b \cdot c$
(2) $\lambda (a \cdot b) = (\lambda a) \cdot b$
(3) $a \cdot b = b \cdot a$
(4) $a \cdot a > 0$ if $a \neq 0$
(5) $a \cdot a = 0$ if $a = 0$
Matrices

1. A matrix is a rectangular array of elements arranged in m rows and n columns. It is written as

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} = [a_{ij}]_{m \times n} = A_{m \times n}
\]

a. if \( m = n \), we have a square matrix.
b. if \( m \neq n \), we have a rectangular matrix.
c. principal diagonal consists of all elements having the same row- and column subscripts.
d. diagonal matrix is a matrix with all off-diagonal elements being zero and is square matrix.
e. identity matrix - a diagonal matrix with all ones along the principle diagonal:

\[
\begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
\end{bmatrix} = I_{n \times n}
\]
f. transpose of a matrix:

\[
A = \begin{bmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{m1} & \cdots & a_{mn}
\end{bmatrix} \quad A' = \begin{bmatrix}
  a_{11} & \cdots & a_{m1} \\
  \vdots & \ddots & \vdots \\
  a_{1n} & \cdots & a_{mn}
\end{bmatrix}
\]

2. Two matrices \( A \) and \( B \), which have the same number of rows and columns, are said to be equal \( (A=B) \) if \( a_{ij} = b_{ij} \) for each \( i \) and \( j \).

3. Addition - assuming \( A \) and \( B \) have the same number of rows and columns, then
\[
A + B = \begin{bmatrix}
  a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn}
\end{bmatrix}
\]

1. \( A + B = B + A \)
2. \( A + (B + C) = (A + B) + C \)
3. \( A + O = A \)
4. \( \lambda (A + B) = \lambda A + \lambda B \)

Multiplication: Given \( A_{m \times n} \) and \( B_{n \times q} \), their product is \( C_{m \times q} \) in which each element \( c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \), \( i = 1, \ldots, m \) and \( j = 1, \ldots, q \). The operation is defined only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

1. \( AB \) is read as \( A \) is being post-multiplied by \( B \)
2. \( BA \) is read as \( A \) is being pre-multiplied by \( B \)
3. \( AB \neq BA \) in general
4. Associative property \( (AB)C = A(BC) \)
5. Distributive property \( A(B + C) = AB + AC \)
6. Other properties of matrix multiplication
   \( \begin{array}{l}
   (1) \ I A = AI = A \\
   (2) \ \lambda (AB) = \lambda A B = A (\lambda B) \\
   (3) \ \lambda IA = \lambda A = A \lambda I = A \lambda \\
   (4) \ OA = AO = O
   \end{array} \)

7. The transpose of the product of two matrices \( A \) and \( B \) is the product of the two transposes in reverse: \( (AB)' = B'A' \)

5. Inversion: Given a matrix \( A \), if there exists a matrix, to be denoted by \( A^{-1} \), such that \( AA^{-1} = A^{-1}A = I \), matrix \( A^{-1} \) is called the inverse matrix of \( A \). If \( A^{-1} \) exists, the matrix \( A \) is said to be nonsingular. If \( A^{-1} \) does not exist, the matrix \( A \) is said to be singular. It follows...
from the definition that the operation of inversion is defined only for square matrices.

a. determining $A^{-1}$ is time consuming, but can be done with elementary row operations. Another way will be noted below.

b. the inverse of $A$, when it exists, satisfies the following properties:

1. $A^{-1}$ is unique.
2. $(A^{-1})^{-1} = A$
3. if $AB = 0$ and $A$ is nonsingular, then $B = 0$

6. powers of matrices - consider a square, nonsingular matrix $A$, then

(i) $A^n A^m = A^{n+m}$
(ii) $(A^n)^m = A^{nm}$

7. partitioning of matrices - given any $m \times n$ matrix $A$, it can be partitioned into various subsets of its elements. A subset matrix is any matrix made up of $h$ rows and $k$ columns belonging to $A$ and such that $h \leq m$ and $k \leq n$, where at least one of the two inequalities is a strict inequality. An example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \cdots A_{22}$ are subsets of $A$

a. addition - $A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$ this assumes $A$ and $B$ are conformably partitioned for the operation of addition.

b. multiplication - assuming that $A$ and $B$ have been suitably partitioned for multiplication, we have
\[
\begin{bmatrix}
A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\
A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22}
\end{bmatrix}
\]

\[
A^{-1} = \left( A_{11} - A_{12}A_{22}^{-1}A_{21} \right)^{-1}
\left( A_{11} - A_{12}A_{22}^{-1}A_{21} \right)^{-1}A_{21}A_{22}^{-1}
\]

\[
= A_{22}^{-1}A_{21}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{21}A_{22}^{-1}
\]

Let us now consider a special kind of partition matrix which will be of some use later on. Consider a partitioned matrix where \(A_{11}\) and \(A_{22}\) are square submatrices, while \(A_{12}\) and \(A_{21}\) are rectangular submatrices. Now if \(A_{12} = 0\), we have a lower block-triangular matrix while if \(A_{21} = 0\) we have an upper block-triangular matrix.

(i) if \(A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\), then \(A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{12}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}\)

(ii) if \(A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\), \(A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}\)

8. Determinants - given any square matrix \(A\), there exists a scalar associated with it which is called the determinant of \(A\). The determinant of the \(n\)th-order square matrix \(A\) is a number, written \(|A|\), obtained by the operation \(|A| = \sum (\pm) a_{1p}a_{2q}a_{3r}...a_{nn}\), where the sum is over all possible permutations of the second (i.e., column) subscripts \(p, q, r, ..., s\). Each addendum has the sign (+) if it refers to an even permutation or the sign (-) if it refers to an odd permutation.

(i) the determinant of a 2x2 matrix is \(|A| = a_{11}a_{22} - a_{12}a_{21}\)

(ii) the determinant of a 3x3 matrix is:

\(|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})\)
(3) Determinant of a diagonal and triangular matrix is given by the product of the elements on its principal diagonal.
(4) The determinant of a block-triangular matrix is given by the product of the determinants of the square submatrices on its principal diagonal.

b. Properties of determinants
(1) \( |A| = |A'| \)
(2) Let \( B \) be a matrix obtained from matrix \( A \) by
   (a) multiplying a row (column) of \( A \) by a scalar \( \lambda \), then \( |B| = |A| \lambda \)
   (b) interchanging two rows (columns) of \( |A| \), then \( |B| = -|A| \)
   (c) adding a multiple of a row (column) of \( A \) to another, then \( |B| = |A| \)
(3) If \( |A| = 0 \), then \( A \) is singular and \( A^{-1} \) does not exist.
(4) If \( |A| \neq 0 \), then \( A \) is nonsingular and \( A^{-1} \) exists.
(5) \( |\lambda A| = \lambda^n |A| \)
(6) \( |AB| = |A| |B| \)

9. Rank - The rank of a matrix gives the maximum number of independent rows (or columns): \( r(A_{m \times n}) \leq \min(m, n) \). The rank of a matrix can be determined via elementary row operations.
   a. If a square matrix \( n \times n \) has a rank of \( n \) then it is nonsingular.
   b. If \( A \) and \( B \) are two matrices, conformable for multiplication, then \( r(AB) \leq \min[r(A), r(B)] \)
   c. Given a matrix \( A \) and two nonsingular matrices \( B \) and \( C \), conformable to \( A \) for post- and pre-multiplication, respectively, then \( r(AB) = r(AC) = r(A) \)
C. Systems of Linear Equations

A linear equation is one involving only the first powers of the unknown variables. An example would be \( ax + c = b \). It derives its name from the fact that the set of points satisfying a linear equation in two variables is a straight line in 2-dimensional space. For a linear equation in \( n \) variables this set will be a hyperplane in \( n \)-dimensional space. A system of linear equations is a set of equations involving a given set of variables—an example would be

\[
\begin{align*}
 a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
 \vdots & \quad \vdots & \quad \vdots \\
 a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{align*}
\]

or

\[
A_{m \times n} x_{n \times 1} \overset{\text{T}}{=} b_{m \times 1}, \quad \text{or} \quad A x^\text{T} = b^\text{T}
\]

2. To determine whether a solution for \( x \) exists, let us first consider the concept of the augmented matrix \( (A, b) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \)

a. if \( r(A, b) > r(A) \), the system has no solution
b. if \( r(A, b) = r(A) = n \)—the number of unknowns, the system has a unique solution

c. if \( r(A, b) = r(A) = k \), where \( k \) is the number of linearly independent equations and if the number of unknowns, \( n \), is greater than \( k \) (i.e. \( k < n \)), then the number of solutions is infinite.
2. The system of simultaneous linear equations: \( Ax = b \) is said to be homogeneous if \( b = 0 \). Otherwise, the system is non-homogeneous.

a. The rank of the augmented matrix of a homogeneous system is always equal to the rank of the coefficient matrix, i.e., \( r(A, \mathbf{0}) = r(A) \).

(i) If \( r(A) = n \), the system has a unique solution - the trivial solution \( x = 0 \). In this case, the number of equations equals the number of unknowns.

(ii) If \( r(A) = k \) and if \( k < n \), the system will have as a solution the trivial solution plus an infinite number of non-trivial solutions.

b. Non-homogeneous systems

(i) The system will have a unique solution if and only if \( r(A) = r(A, b) = n \), where \( n \) is the number of variables.

(ii) The system will have infinite solutions if \( r(A) = r(A, b) = k < n \).

4. Eigenvectors and Eigenvalues

a. Let us consider the following situation - given a square matrix \( A \), let us find those vectors \( x \neq 0 \) such that \( Ax = \lambda x \) where \( \lambda \) is an eigenvalue of \( A \) and \( x \) is an eigenvector of \( A \) corresponding to \( \lambda \).

b. To solve for the eigenvalues of \( A \), the following steps are taken:

(1) Rewrite \( Ax = \lambda x \) as \( A x - \lambda I x = 0 \) or \( (A - \lambda I)x = 0 \) which happens to be a homogeneous system.

(2) Since \( x \neq 0 \), the necessary and sufficient condition for a non-zero solution to exist is that \( r(A - \lambda I) < n \) or

\[
|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0
\]
This algebraic equation is called the characteristic equation of matrix $A$.

(3) solving the characteristic equation, we find that it has $n$ solutions or roots, denoted as $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are called the eigenvalues of matrix $A$. The eigenvalues may be complex or real numbers and may all be distinct or there may be duplication.

To determine the eigenvectors associated with each eigenvalue, the following steps are taken:

1. Selecting a particular eigenvalue $\lambda^*$, plug it into $(A - \lambda^*I)x = 0$.
2. Since $\text{r}(A - \lambda^*I) < n$, there exists more unknowns than equations; therefore to solve for $x$ the "excess" unknowns must be assigned, arbitrarily, values.
3. If $\text{r}(A - \lambda^*I) = n - 1$, then one component of $x$ is assigned a value and the other $n - 1$ unknowns are solved for in terms of it. If the number of excess unknowns is more than one, then each one must be assigned a value before solving for the rest.

Properties of eigenvalues and eigenvectors

1. The eigenvalues of $A$ and $A'$ are identical.
2. $Ax = \lambda x - x$ is called a right-hand eigenvector (RHEV).
3. $yA = \lambda y - y$ is called a left-hand eigenvector (LHEV).
4. Given a square matrix $A$, if its $n$ eigenvalues are all distinct, matrix $A$ has $n$ linearly independent eigenvector's equations.

5. $A^n x_\lambda = \lambda^n x_\lambda$
6. $(I - A)x_\lambda = (I - \lambda)x_\lambda$
7. $(I - (1+r)A)x_\lambda = (\lambda - (1+r)\lambda)x_\lambda$
5. Convergence conditions for a square matrix and bounds to eigenvalues

a. Consider the powers of a square matrix $A$, namely $I, A, A^2, A^3, \ldots$. Matrix $A$ is said to be a convergent matrix if $\lim_{n \to \infty} A^n = O$.

Otherwise, it is said to be a non-convergent matrix. Observe that, for matrix $A^n$ to tend to the null matrix as $n$ increases, it is necessary for all the elements of $A$ to tend to zero as $n \to \infty$. The necessary and sufficient condition for this to occur is that all the eigenvalues have an absolute value of less than one, i.e., $|\lambda_m| < 1$ where $\lambda_m$ is the maximum eigenvalue.

b. To determine whether a matrix converges to the null matrix, the boundary conditions of eigenvalues need to be delineated. To do so, the concept of matrix norm needs to be introduced. A matrix norm can be defined in two ways:

(i) $\|A\|_1 = \max \sum_{j=1}^n |a_{ij}|$ - the matrix norm is the maximum absolute row sum

(ii) $\|A\|_2 = \max \sum_{i=1}^n |a_{ij}|$ - the matrix norm is the maximum absolute column sum

(iii) Example:

\[
A = \begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{bmatrix}
\]

$\|A\|_1 = \max \sum_{j=1}^n |a_{1j}| + \cdots + |a_{nj}|$

$\|A\|_2 = \max \sum_{i=1}^n |a_{ij}| + \cdots + |a_{in}|$

a. To relate matrix norms to eigenvalues, we find that the maximum eigenvalue is bounded by the column or row norm of $A$.
(i) \( \| A \|_1 = \max \sum_{j=1}^{n} |a_{ij}| \geq |\lambda_m| \)

(ii) \( \| A \|_\infty = \max \sum_{i=1}^{n} |a_{ij}| \geq |\lambda_m| \)

Therefore if the matrix norm is less than one, then \( |\lambda_m| < 1 \) and, consequently \( A^n \to 0 \).

b. The geometric series \( 1 + A + A^2 + A^3 + \cdots \) converges iff \( A \) is convergent. Moreover, if \( A \) is convergent, the \( I - A \) is non-singular and \( (I - A)^{-1} = I + A + A^2 + A^3 + \cdots \). This result can be extended in the following manner - given a real number \( v \) and \( A \), we have:

1. \( (vA)^n \to 0 \) if and only if \( |\lambda_m| < 1 \) and \( v < 1 \) or if \( v < \frac{1}{|\lambda_m|} \)
2. The geometric series \( 1 + vA + v^2A^2 + v^3A^3 + \cdots \) converges if \( |\lambda_m| < 1 \) and \( v < \frac{1}{|\lambda_m|} \)
3. If \( vA \) is convergent, then \( I - vA \) is non-singular and \( (I - vA)^{-1} = I + vA + v^2A^2 + v^3A^3 + \cdots \).

### Systems of Linear Equations and Non-Negative Matrices

Consider the following system of \( n \) equations \( Ax = b \):

a. \( A \) will be called non-negative iff every element \( a_{ij} \geq 0 \)

b. \( A \) will be called semi-positive iff every row and column of \( A \) is a semi-positive vector, i.e. every \( a_{ij} \geq 0 \) and at least one element of each row and column is strictly positive.

c. \( A \) will be called positive iff every element \( a_{ij} > 0 \)

In order to discuss the properties of non-negative matrices we need to introduce a fundamental concept relating to the structure of a
matrix. That is we must discover whether the rows and columns of the matrix can be reorganized to form a block of zeros.

1. A square matrix $A$ is said to be decomposable if it can written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \text{ or } A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

If the matrix cannot be reordered in his form, then it is indecomposable. (The terms reducible, irreducible are sometimes used instead of decomposable, indecomposable.)

2. In terms of graph theory, consider $n$ points $1, \ldots, n$. For each $i$ and $j$ such that $a_{ij} \neq 0$, draw an arrow from point $i$ to point $j$. If we can get from each element to any other element by a sequence of such arrows, then $A$ is irreducible - example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

If by continued permutation of rows and columns the matrix $A$ can be reordered to give a matrix of the form: $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ where $A_{11}$ and $A_{22}$ square and indecomposable, then $A$ is completely decomposable.

3. Perron-Frobenius theorems concerning indecomposable non-negative square matrices

a. Let $A \geq 0$ be an indecomposable semi-positive $n$th-order square matrix and $x > 0$ an $n$-component vector, the following points can be made:

(i) the vector $z = (I + A)x$ always has a number of zero elements less than that of $x$. 

(2) the $(I+A)^n > 0$ and $A(I+A)^n > 0$

(3) If $A \geq 0$, then $\lambda_m > 0$ and $x(\lambda_m) > 0$ - that is, the maximum eigenvalue of a semi-positive (or positive matrix) is positive and its corresponding eigenvector has all positive elements. The above point relating to the maximum eigenvalue can be shown in the following manner: $\lambda_m$ satisfies the following inequalities:

(i) $\min_j \sum_{i=1}^n a_{ij} \leq \lambda_m \leq \max_j \sum_{i=1}^n a_{ij}$ - that is, $\lambda_m$ falls between the minimum row-sum and the maximum row-sum (or matrix norm).

(ii) $\min_i \sum_{j=1}^n a_{ij} \leq \lambda_m \leq \max_i \sum_{j=1}^n a_{ij}$ - that is, $\lambda_m$ falls between the minimum column-sum and the maximum column-sum (or matrix norm).

(iii) Since $A$ is semi-positive, $\lambda_m$ is bounded from below and above by positive numbers thus ensuring that it is a positive number.

(4) Let $B$ and $A$ be semi-positive indecomposable square matrices, then if $\lambda_m B = \lambda_m A$, $B = A$.

(5) $\lambda_m$ of $A$ is a continuous, increasing function of the elements of $A$. That is, $\lambda_m$ increases involve as the elements of $A$ increase involve. Two corollaries follow from this:

(i) The maximum eigenvalue of any square submatrix of $A$ is smaller than the maximum eigenvalue $\lambda_m$ of $A$. 
(iii) the eigenvalue \( \lambda_m \) is a simple root of the characteristic equation of \( A \).

(6) to each real eigenvalue \( \lambda \) of \( A \) different from \( \lambda_m \) there corresponds an eigenvector \( x \neq 0 \) which has at least one negative component.

(7) given a real number \( \mu = \left( \frac{1}{v} \right) > 0 \), if \( \mu > \lambda_m \) and \( v < \sqrt{\lambda_m} \), then \((\mu I - A)^{-1} > 0 \) and \((I - vA)^{-1} > 0 \). That is, all the elements of both matrices are positive real numbers. Moreover, all the elements of the matrices are continuous, increasing functions of \( v \) and continuous, decreasing functions of \( \mu \).

The above points collectively state the essential properties of the Perron-Frobenius theorem which can be stated as follows:

Any indecomposable semi-positive square matrix \( A \) possesses a positive eigenvalue \( \lambda_m \) and a positive eigenvector \( x \) corresponding to \( \lambda_m \). \( \lambda_m \), the only eigenvalue of \( A \) with an associated positive eigenvector, is a simple root of the characteristic equation \( |A - \lambda I| = 0 \). The absolute values of the \( n-1 \) other eigenvalues of \( A \) are not greater than \( \lambda_m \).

4. properties of semi-positive decomposable square matrices

a. \( \lambda_m \) is associated with a non-negative eigenvector.

b. \( \lambda_m \) is a continuous non-decreasing function of the elements of \( A \).

c. \( \lambda_m \) of any square submatrix of \( A \) cannot be greater than the \( \lambda_m \) of \( A \).

d. \( \lambda_m \) is not necessarily a simple root of \( |A - \lambda I| \).
e. given a real number $\mu = (\frac{1}{v}) > 0$, if $\mu > \lambda_m$ and $v < \frac{1}{\lambda_m}$, then $(\mu I - A)^{-1} \geq 0$ and $(I - \nu A)^{-1} \geq 0$. That is, all the elements of the above matrices are non-negative real numbers. Moreover, all the elements of the matrices are continuous non-decreasing functions of $\nu$ and continuous, non-increasing functions of $\mu$.

E. Solving Specific Kinds of Linear Equations with Non-Negative Matrices

1. For pedagogical purposes, a indecomposable square matrix will be will be assumed and used.

2. Let us first consider the following system of equations $Ax = x\lambda$ or $(A - \lambda I)x = 0$. In this case we are dealing with a homogeneous system of equations; therefore if a non-trivial solution exists, $(A - \lambda I)x = 0$. Assuming that there are $n-1$ independent rows (or columns), then the solution for $x$ (that is $x$ is a vector of unknowns) is found by first solving the characteristic equation; second, taking any particular eigenvalue and substituting it into $(A - \lambda I)x = 0$; third, assuming an arbitrary value for say $x_1$; and lastly, solving for $x_i$ the eigenvector associated with $\lambda_i$, in terms of $x_1$.

3. Now let us consider a second system of equations $Ax + F = x$ where $x$ is a vector of $n$ unknowns and $F$ is a vector whose components are non-negative and constant. There are three principle ways of solving for $x$ - the Gaussian elimination method, iterative
method, and inversion.

2. Gaussian elimination method

(1) To use this method, the above equation must be rewritten in the following manner: \(Ax + f = x \rightarrow x - Ax = f \rightarrow (I - A)x = f\) or

\[(1-a_{11})x_1 - \cdots - a_{1n}x_n = f_1\]
\[\cdots\cdots\cdots\cdots\cdots\]
\[-a_{nn}x_n = (1-a_{nn})x_n = f_n\]

(2) To solve this, we subtract \(a_{ii}/1-a_{ii}\) times the first equation from the \(i\)th equation to get the derived system

\[(1-a_{ii})x_1 - \cdots - a_{in}x_n = f_1\]
\[0 - a_{i2}x_2 - \cdots - a_{in}x_n = f_2\]
\[\cdots\cdots\cdots\cdots\cdots\]
\[0 - a_{n2}x_2 - \cdots - (1-a_{nn})x_n = f_n\]

(3) This procedure is maintained until all \(n\) equations are dealt with and thus arriving at a triangularized system of the following form:

\[(1-a_{ii})x_1 - a_{i2}x_2 - \cdots - a_{in}x_n = f_1\]
\[0 - a_{i2}x_2 - \cdots - a_{in}x_n = f_2\]
\[\cdots\cdots\cdots\cdots\cdots\]
\[0 \cdots 0 - a_{n(n-1)}x_{n-1} - a_{nn}x_n = f_{n-1}\]

(4) To calculate the solution, back substitution is used.

(5) A variation of this approach is called Gauss-Jordan reduction. The Gaussian elimination method is used, but with a twist in that all non-diagonal elements are reduced to zero.

(6) Another variant is called the Crout reduction.
iterative method

(1) In the capacity of an initial approximation, let us take a certain vector \( x^0 \), chosen, generally speaking, quite arbitrarily. Next, let us construct the vectors
\[
A x^0 + f = x^1 \\
A x^1 + f = x^2 \\
\ldots \\
A x^{k-1} + f = x^k
\]
If the sequence \( x^0, x^1, x^2, \ldots \), approaches the limit where \( x^{k+1} \approx x^k \), then we can say that the method of iteration will produce a solution for \( x \).

(2) For the iterative process to converge, \( \lambda_m \) of \( A \) has to be less than one. This can be seen by rearranging the above equations in the following manner:
\[
A^k x^0 + (I + A + A^2 + \ldots + A^{k-1}) f = x^k
\]
So as \( k \to \infty \) the iterative process will converge (i.e., \( A^k \to 0 \)), if and only if \( \lambda_m < 1 \) (see page 13).

(3) The above iterative method is a special form of the Jacobi's iterative method. Dividing a matrix into three parts, one part being a lower triangular matrix with zeros along the main diagonal, a second being a upper triangular matrix with zeros along the main diagonal and the third being a diagonal matrix, we have
\[
J = (I + D + L + U) = (D + U + L) = (I - A)
\]
where \( J \) is the Jacobi matrix;
\[
D + L + U = A; \quad \text{and}
\]
$D^*$ is the diagonal matrix.

Working from the standard form $Jx = f$ we have the following:

$$Jx = f \Rightarrow$$

$$(I - D - L - U)x = f$$

$Ix = (D + L + U)x + f$

$$x = A_x + f$$

c. Inversion method

(i) Starting with the form $A_x + f = x$ we have:

$$f = x - A_x \Rightarrow$$

$$f = (I - A)x \Rightarrow$$

$$(I - A)^{-1}f = x$$

(ii) For this method to work, $A$ must be nonsingular. That is since $I$ is nonsingular, $(I - A)$ will be nonsingular if $A$ is nonsingular.

Let us now compare the solutions, in general, that are derived from the iterative and inversion method.

a. As noted above (page 14), $(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$ if $\lambda_m < 1$. Therefore the solution vector under the inversion method can be written as $$(I + A + A^2 + \cdots)f = x$$ if $\lambda_m < 1$. Comparing this to the iterative method, the two solution vectors converge as $k \to \infty$. Therefore when $\lambda_m < 1$ the iterative and inversion methods produce the same solution vector.

Let us now take a closer look at the solution vector $x$. As noted on page 17, $(I - A)^{-1} > 0$ if $\lambda_m < 1$. This is easily seen since $(I - A)^{-1} = I + A + A^2 + \cdots$ because $I, A, A^2, \text{ etc.}$ are all semi-positive matrices. Therefore as long as $f$ is a semi-positive vector, the
solution vector will be strictly positive.

Since the value of the maximum eigenvalue of $A$ is significant with respect to the solution vector, a more detailed discussion is necessary. As indicated on page 16, $\lambda_m < 1$ if none of the rows or columns add up to one. This restriction can be weakened in the following manner:

If $A$ is a semi-positive, indecomposable matrix none of whose column or row sums is greater than one, and at least one of which is less than one, then $\lambda_m < 1$. Proof (for column norm)—
given the boundary condition of $\lambda_m(A)$ as

$$\min_j \sum_{i=1}^{n} a_{ij} \leq \lambda_m = \max_j \sum_{i=1}^{n} a_{ij},$$

we want to prove that $\lambda_m$ equals the maximum (and minimal) column sum if all are equal. Now assume that the maximum equality holds, but not all columns are equal. Then by increasing the positive elements of $A$, we produce a new matrix $A^*$ in which all columns are equal and equal one. Since $\lambda_m(A^*) = 1, \lambda_m(A) < 1$; if this was not the case, i.e. $\lambda_m(A^*) = \lambda_m(A)$, then $A^* = A$, but this contradicts the above statement that the positive elements of $A$ were increased. The importance of this proof will become evident later.

Let us finally consider the following set of equations

$$(1+r)A\mathbf{x} + \mathbf{f} = \mathbf{x},$$

where $\mathbf{x}$ is a given positive real number.

1. Solving for $\mathbf{x}$ by the inversion method:

$$(I-(1+r)A)^{-1}\mathbf{f} = \mathbf{x}$$

2. If $1+r = v < \frac{1}{\lambda_m}$, then $(I-(1+r)A)^{-1}$ will converge and $\mathbf{x} > 0$.

3. If $\lambda_m$ increases, then $1+r$ decreases.
F. Problems

1. Let \( A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \), \( B = \begin{bmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{bmatrix} \), \( C = \begin{bmatrix} 2 & -3 & 0 \\ 5 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \), \( D = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \)

Find (i) \( A + B \) (vi) \( AD \) (xi) \( A' \cdot C \)

(ii) \( A + C \) (vii) \( BC \) (xii) \( D' \cdot A' \)

(iii) \( 3A - 4B \) (viii) \( BD \) (xiii) \( B' \cdot A \)

(iv) \( AB \) (ix) \( CB \) (xiv) \( D' \cdot D \)

(v) \( AC \) (x) \( A' \) (xv) \( D \cdot D' \)

2. Square Matrices, Determinants and Inverses

a. Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \). Find (i) \( A^2 \), (ii) \( |A| \), (iii) \( A^{-1} \), (iv) \( \text{rank of } A \)

b. Let \( A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} \). Find (i) \( A^2 \), (ii) \( |A| \), (iii) \( A^{-1} \), (iv) \( \text{rank of } A \)

c. Let \( A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \). Find (i) \( A^2 \), (ii) \( |A| \), (iii) \( A^{-1} \), (iv) \( \text{rank of } A \)

3. Eigenvalues and Eigenvectors

\( A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix} \), \( B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \), \( C = \begin{bmatrix} 5 & 4 \\ 5 & 3 \end{bmatrix} \)

For \( A, B, C \), find the characteristic equation, the eigenvalues, and the eigenvectors associated with each eigenvalue.
4. Linear Systems

For each of the following linear systems, solve for $x$.

a. \[
\begin{pmatrix}
0.6 & 0.3 \\
0.4 & 0.7
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} =
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
\text{ solve for } \lambda_m(x)
\]

b. \[
\begin{pmatrix}
0.2 & 0.3 \\
0.8 & 0.7
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} =
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
\text{ solve for } \lambda_m(x)
\]

c. \[
\begin{pmatrix}
1.2 & 0.1 \\
0.7 & 0.5
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} =
\begin{bmatrix}
3 \\ 2
\end{bmatrix}
\]

d. \[
\begin{pmatrix}
0.2 & 0.3 \\
0.4 & 0.2
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} +
\begin{bmatrix}
0.5 \\ 0.5
\end{bmatrix} =
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
\]

e. \[
\begin{pmatrix}
280 & 12 \\
120 & 8
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} =
\begin{bmatrix}
400 x_1 \\ 20 x_2
\end{bmatrix}
\text{ solve for } \lambda_m(x)
\]

f. \[
\begin{pmatrix}
280 & 12 \\
120 & 8
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} (1+r) =
\begin{bmatrix}
575 x_1 \\ 20 x_2
\end{bmatrix}
\text{ (hint } \lambda_m = 1/1+r)
\]

g. \[
\begin{pmatrix}
0.6 & 0.3 \\
0.3 & 0.2
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} +
\begin{bmatrix}
3 \\ 2
\end{bmatrix} =
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
\]

h. \[
\begin{pmatrix}
2.4 & 0.7 \\
5.2 & 0.7
\end{pmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} (1+r) +
\begin{bmatrix}
3 w \\ 7 w
\end{bmatrix} =
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
\text{ (assume } w=1 \text{ and } r = 25\%)
\]
i. \[
\begin{bmatrix}
0.4 & 0.6 \\
0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
(1 + r) + \begin{bmatrix} 1 \\
1
\end{bmatrix} w = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]
assume \(x_1 = 1\) and \(r = 10\%\)

j. \[
\begin{bmatrix}
20 & 0 \\
3 & 10
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
(1 + r) + \begin{bmatrix} 23 \\
11
\end{bmatrix} w = \begin{bmatrix}
23 x_1 \\
11 x_2
\end{bmatrix}
\]
assume \(w = 1\) and \(r = 5\%\)

G. Answers

1. (i) \[
\begin{bmatrix}
5 & -1 & -1 \\
-1 & 1 & 7
\end{bmatrix}
\]
(ii) matrices
(iii) \[
\begin{bmatrix}
-13 & -3 & 18 \\
4 & 17 & 0
\end{bmatrix}
\]

(iv) matrices
(v) \[
\begin{bmatrix}
-5 & -2 & 4 & 5 \\
11 & -3 & -12 & 18
\end{bmatrix}
\]
(vi) \[
\begin{bmatrix}
9
\end{bmatrix}
\]

(vii) \[
\begin{bmatrix}
11 & -12 & 0 & -5 \\
-15 & 5 & 8 & 4
\end{bmatrix}
\]
(viii) \[
\begin{bmatrix}
-1 \\
9
\end{bmatrix}
\]
(ix) matrices

(xi) \[
\begin{bmatrix}
10 \\
-1 \\
3 \\
24
\end{bmatrix}
\]

(xii) \[
\begin{bmatrix}
9 \\
9
\end{bmatrix}
\]

2. (a) (i) \[
\begin{bmatrix}
7 & 10 \\
15 & 22
\end{bmatrix}
\]
(ii) \[
\begin{bmatrix}
-2 \\
3/2 \\
22
\end{bmatrix}
\]
(iii) \[
\begin{bmatrix}
-2 \\
1
\]
\[
\begin{bmatrix}
3/2 \\
-1/2
\end{bmatrix}
\]
(iv) \[
2
\]
b. (i) \[ \begin{bmatrix} 19 & 30 \\ 12 & 19 \end{bmatrix} \] (iii) -1 (iii) \[ \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} \] (iv) 2

c. (iii) \[ \begin{bmatrix} 33 & 44 \\ 66 & 88 \end{bmatrix} \] (ii) 0 (iii) does not exist (iv) 1

3. a. (i) \( \lambda^2 + \lambda - 11 \) (iii) \( \lambda_{1,2} = \frac{1}{2} \pm \sqrt{45}/2 \)
(c.iii) \( \lambda_1: (1, 1.17) \quad \lambda_2: (1, -1.17) \)

b. (i) \( \lambda^2 - 5\lambda - 14 \) (ii) \( \lambda_1 = 7, \lambda_2 = -2 \)
(iii) \( \lambda_1: (1, 1) \quad \lambda_2: (1, -1.25) \)

c. (i) \( \lambda^2 - 8\lambda - 0.05 \) (ii) \( \lambda_1 = 8.5826, \lambda_2 = -0.05826 \)
(iii) \( \lambda_1: (1, 0.9) \quad \lambda_2: (1, -1.395) \)

4. a. \( \lambda_m = 1 \) so \( \lambda_m: (1, 4/3) \)

b. \( \lambda_m = 1 \) so \( \lambda_m: (1, 8/3) \)

c. \( x_1 = 43.33 \quad x_2 = -56.67 \)

d. \( x_1 = 1.06 \quad x_2 = 1.15 \)

e. \( \lambda_m = 1 \) so \( \lambda_m: (1, 10) \)

f. \( \lambda_m = 8 \) so \( r = 1.25 \) \( X_1 = 1 \quad X_2 = 15 \)

g. \( X_1 = 1.219 \bar{Y}_1 + 0.732 \bar{Y}_2 \)
\( X_2 = 0.366 \bar{Y}_1 + 1.219 \bar{Y}_2 \)

h. \( x_1 = 2.65 \quad x_2 = 2.7 \)

i. \( \omega = 0.009 \quad x_2 = 34.05 \)

j. \( x_1 = 1.913 \quad x_2 = 34.05 \)
REFERENCES AND READINGS

A. Vectors


B. Matrices


C. Systems of Linear Equations


D. Systems of Linear Equations and Non-Negative Matrices


E. Solving Specific Kinds of Linear Equations with Non-Negative Matrices