7.2 Derivations of the Theorems

Note for this section: I used slightly different notation here than Copi does in the text. I went with the modern symbols for the empty and universal sets, Ø and **U**, respectively.

1.) There is at most one entity Ø in **C** such that $\alpha \cup \varnothing = \alpha$

Suppose there are entities $\varnothing_1$ and $\varnothing_2$ in **C** such that $\alpha \cup \varnothing_1 = \alpha$ and $\alpha \cup \varnothing_2 = \alpha$

1. $\varnothing_1 \cup \varnothing_2 = \varnothing_1$	Ax. 3
2. $\varnothing_2 \cup \varnothing_1 = \varnothing_2$	Ax. 3
3. $\varnothing_1 \cup \varnothing_2 = \varnothing_2 \cup \varnothing_1$	Ax. 5
4. $\varnothing_1 = \varnothing_2 \cup \varnothing_1$	Substitution (1 into 3)
5. $\varnothing_1 = \varnothing_2$	Substitution (2 into 4)

3.) $\alpha \cup \alpha = \alpha$

1. $\alpha \cup \varnothing = \alpha$	Ax. 3
2. $\alpha \cap \bar{\alpha} = \varnothing$	Ax. 9
3. $\alpha \cup (\alpha \cap \bar{\alpha}) = \alpha$	Substitution (2 into 1)
4. $(\alpha \cup \alpha) \cap (\alpha \cup \bar{\alpha}) = \alpha$	Ax. 7 (applied to left side of 3)
5. $(\alpha \cup \alpha) \cap \mathbf{U} = \alpha$	Ax. 9 [applied to $(\alpha \cup \bar{\alpha})$ in 4]
6. $(\alpha \cup \alpha) \cap \mathbf{U} = \alpha \cup \alpha$	Ax. 4
7. $\alpha \cup \alpha = \alpha$	Substitution (6 into 5)

4.) $\alpha \cap \alpha = \alpha$

1. $\alpha \cap \mathbf{U} = \alpha$	Ax. 4
2. $\alpha \cup \bar{\alpha} = \mathbf{U}$	Ax. 9
3. $\alpha \cap (\alpha \cup \bar{\alpha}) = \alpha$	Substitution (2 into 1)
4. $(\alpha \cap \alpha) \cup (\alpha \cap \bar{\alpha}) = \alpha$	Ax. 8 (applied to left side of 3)
5. $(\alpha \cap \alpha) \cup \varnothing = \alpha$	Ax. 9 [applied to $(\alpha \cap \bar{\alpha})$ in 4]
6. $(\alpha \cap \alpha) \cup \varnothing = \alpha \cap \alpha$	Ax. 3
7. $\alpha \cap \alpha = \alpha$	Substitution (6 into 5)

5.) $\alpha \cup \mathbf{U} = \mathbf{U}$

1. $\bar{\alpha} \cap \mathbf{U} = \bar{\alpha}$	Ax. 4
2. $\alpha \cup \bar{\alpha} = \mathbf{U}$	Ax. 9
3. $\alpha \cup (\bar{\alpha} \cap \mathbf{U}) = \mathbf{U}$	Substitution (1 into 2)
4. $(\alpha \cup \bar{\alpha}) \cap (\alpha \cup \mathbf{U}) = \mathbf{U}$	Ax. 7 (applied to left side of 3)
5. $\mathbf{U} \cap (\alpha \cup \mathbf{U}) = \mathbf{U}$	Substitution (2 into 4)
6. $(\alpha \cup \mathbf{U}) \cap \mathbf{U} = \mathbf{U}$	Ax. 6 (applied to left side of 5)
7. $(\alpha \cup \mathbf{U}) \cap \mathbf{U} = \alpha \cup \mathbf{U}$	Ax. 4
8. $\alpha \cup \mathbf{U} = \mathbf{U}$	Substitution (7 into 6)

6.) $\alpha \cap \emptyset = \emptyset$

1. $\bar{\alpha} \cup \emptyset = \bar{\alpha}$	Ax. 3
2. $\alpha \cap \bar{\alpha} = \emptyset$	Ax. 9
3. $\alpha \cap (\bar{\alpha} \cup \emptyset) = \emptyset$	Substitution (1 into 2)
4. $(\alpha \cap \bar{\alpha}) \cup (\alpha \cap \emptyset) = \emptyset$	Ax. 8 (applied to left side of 3)
5. $\emptyset \cup (\alpha \cap \emptyset) = \emptyset$	Substitution (2 into 4)
6. $(\alpha \cap \emptyset) \cup \emptyset = \emptyset$	Ax. 5 (applied to left side of 5)
7. $(\alpha \cap \emptyset) \cup \emptyset = \alpha \cap \emptyset$	Ax. 3
8. $\alpha \cap \emptyset = \emptyset$	Substitution (7 into 6)

7.) $\emptyset \neq \mathbf{U}$

*Even though Copi presents a proof for this theorem in the back of the text I decided to include an alternate version in keeping with the spirit of avoiding indirect proofs.

Suppose $\emptyset = \mathbf{U}$. By Ax. 10 there exists an α in **C** such that $\alpha \neq \emptyset$ and $\alpha \neq \mathbf{U}$.

1. $\emptyset = \mathbf{U}$	Assume
2. $\exists \alpha \in \mathbf{C}$: $\alpha \neq \emptyset$ and $\alpha \neq \mathbf{U}$	Ax. 10
3. $\alpha \cap \mathbf{U} = \alpha \cap \emptyset$	by hypothesis ($\emptyset = \mathbf{U}$)
4. $\alpha = \alpha \cap \emptyset$	Ax. 4 (applied to left side of 3)
5. $\alpha = \emptyset$	Th. 6 (applied to right side of 4)
6. $\alpha \neq \mathbf{U}$	2, Simp
7. $\emptyset \neq \mathbf{U}$	Substitution (5 into 6)
8. $(\emptyset = \mathbf{U}) \rightarrow (\emptyset \neq \mathbf{U})$	1-7 CP
9. $\sim (\emptyset = \mathbf{U}) \vee (\emptyset \neq \mathbf{U})$	8, MI
10. $(\emptyset \neq \mathbf{U}) \vee (\emptyset \neq \mathbf{U})$	Substitution ([$\sim (\emptyset = \mathbf{U}) \equiv (\emptyset \neq \mathbf{U})$] into 9)
11. $\emptyset \neq \mathbf{U}$	10, Tautology

8.) If $\alpha = \overline{\beta}$, then $\beta = \overline{\alpha}$

1. $\alpha = \overline{\beta}$	Assume
2. $\beta \cup \emptyset = \beta$	Ax. 3
3. $\alpha \cap \overline{\alpha} = \emptyset$	Ax. 9
4. $\overline{\beta} \cap \overline{\alpha} = \emptyset$	Substitution (1 into 3)
5. $\beta \cup (\overline{\beta} \cap \overline{\alpha}) = \beta$	Substitution (4 into 2)
6. $(\beta \cup \overline{\beta}) \cap (\beta \cup \overline{\alpha})$	Ax. 7
7. $\mathbf{U} \cap (\beta \cup \overline{\alpha}) = \beta$	Ax. 9 [applied to $(\beta \cup \overline{\beta})$ in 6]
8. $(\beta \cup \overline{\alpha}) \cap \mathbf{U} = \beta$	Ax. 6
9. $\beta \cup \overline{\alpha} = \beta$	Ax. 4
10. $\overline{\alpha} \cup \beta = \beta$	Ax. 5
11. $\overline{\alpha} \cup \emptyset = \overline{\alpha}$	Ax. 3
12. $\beta \cap \overline{\beta} = \emptyset$	Ax. 9
13. $\beta \cap \alpha = \emptyset$	Substitution (1 into 12)
14. $\overline{\alpha} \cup (\beta \cap \alpha) = \overline{\alpha}$	Substitution (13 into 11)
15. $(\overline{\alpha} \cup \beta) \cap (\overline{\alpha} \cup \alpha) = \overline{\alpha}$	Ax. 7
16. $(\overline{\alpha} \cup \beta) \cap (\alpha \cup \overline{\alpha}) = \alpha$	Ax. 5 [applied to $(\overline{\alpha} \cup \alpha)$ in 15]
17. $(\overline{\alpha} \cup \beta) \cap \mathbf{U} = \overline{\alpha}$	Ax. 9 [applied to $(\alpha \cup \overline{\alpha})$ in 16]
18. $(\overline{\alpha} \cup \beta) = \overline{\alpha}$	Ax. 4 (applied to left side of 17)
19. $\beta = \overline{\alpha}$	Substitution (10 into 18)
20. $(\alpha = \overline{\beta}) \rightarrow (\beta = \overline{\alpha})$	1-19, CP

9.) $\alpha = \overline{\overline{\alpha}}$

Proof follows directly from Theorem 8 above. Every α has a supplement by Ax. 9, so let β be the supplement of α.

1. $\beta = \overline{\alpha}$	Ax. 9 (letting β be the supplement of α)
2. $\alpha = \overline{\beta}$	Th. 8 (applied to 1 with a swap of α and β)
3. $\alpha = \overline{\overline{\alpha}}$	Substitution (1 into 2)

10.) If $\alpha \cap \beta \neq \emptyset$, then $\alpha \neq \emptyset$

*The partial substitution in line 4 is valid because the identity relation allows the swapping of equals to be carried out without qualification. (e.g. x = 2 implies both x + x = 4 and x + 2 = 4.)

1. $\alpha \cap \beta \neq \emptyset$	Assume
2. $\alpha = \emptyset$	Assume
3. $\alpha \cap \beta \neq \alpha$	Substitution (2 into 1)
4. $\emptyset \cap \beta \neq \alpha$	Substitution (Partial: 2 into 3)*
5. $\beta \cap \emptyset \neq \alpha$	Ax. 6 (applied to left side of 4)
6. $\emptyset \neq \alpha$	Th. 6 (applied to left side of 5)
7. $\alpha \neq \emptyset$	Identity (7)
8. $(\alpha = \emptyset) \rightarrow (\alpha \neq \emptyset)$	2-7 CP
9. $\sim (\alpha = \emptyset) \text{ v } (\alpha \neq \emptyset)$	8, MI
10. $(\alpha \neq \emptyset) \text{ v } (\alpha \neq \emptyset)$	Substitution ($[\sim (\alpha = \emptyset) \equiv (\alpha \neq \emptyset)]$ into 9)
11. $(\alpha \neq \emptyset)$	10, Tautology
12. $[\alpha \cap \beta \neq \emptyset] \rightarrow [\alpha \neq \emptyset]$	1-11, CP

11.) $\alpha = (\alpha \cap \beta) \cup (\alpha \cap \bar{\beta})$

1. $\alpha \cap \mathbf{U} = \alpha$	Ax. 4
2. $\beta \cup \bar{\beta} = \mathbf{U}$	Ax. 9
3. $\alpha \cap (\beta \cup \bar{\beta}) = \alpha$	Substitution (2 into 1)
4. $(\alpha \cap \beta) \cup (\alpha \cap \bar{\beta}) = \alpha$	Ax. 8 (applied to left side of 3)
5. $\alpha = (\alpha \cap \beta) \cup (\alpha \cap \bar{\beta})$	Identity (4)

13.) $\alpha \cap (\beta \cap \gamma) = (\alpha \cap \beta) \cap \gamma$

*I broke with the order of proofs at this point because I think this proof is drastically simpler and more beautiful if one uses Theorems 17 and 18 to play off the symmetry inherent in the equation. So, keep in mind that Theorems 17 and 18 will be proven without reference to this theorem to avoid any circularity of reasoning.

1. $\alpha \cap (\beta \cap \gamma) = \overline{\overline{\alpha \cap (\beta \cap \gamma)}}$	Th. 9
2. $\overline{\overline{\alpha \cap (\beta \cap \gamma)}} = \overline{\bar{\alpha} \cup (\overline{\beta \cap \gamma})}$	Th. 17
3. $\overline{\bar{\alpha} \cup (\overline{\beta \cap \gamma})} = \overline{\bar{\alpha} \cup (\bar{\beta} \cup \bar{\gamma})}$	Th. 17 (applied to $\overline{\beta \cap \gamma}$)
4. $\overline{\bar{\alpha} \cup (\bar{\beta} \cup \bar{\gamma})} = \overline{(\bar{\alpha} \cup \bar{\beta}) \cup \bar{\gamma}}$	Th. 12
5. $\overline{(\bar{\alpha} \cup \bar{\beta}) \cup \bar{\gamma}} = \overline{\bar{\alpha} \cup \bar{\beta}} \cap \bar{\bar{\gamma}}$	Th. 18
6. $\overline{\bar{\alpha} \cup \bar{\beta}} \cap \bar{\bar{\gamma}} = (\bar{\bar{\alpha}} \cap \bar{\bar{\beta}}) \cap \bar{\bar{\gamma}}$	Th. 18 (applied to $\bar{\alpha} \cup \bar{\beta}$)
7. $(\bar{\bar{\alpha}} \cap \bar{\bar{\beta}}) \cap \bar{\bar{\gamma}} = (\alpha \cap \beta) \cap \gamma$	Th. 9 (applied thrice)
8. $\alpha \cap (\beta \cap \gamma) = (\alpha \cap \beta) \cap \gamma$	Equivalence, 1-7 (To save space I skipped the substitution step which exists between each of these lines, but it is obvious the equivalence holds, as the right side of each line is taken as the left side of each subsequent line.)

14. $\emptyset = \overline{\mathbf{U}}$

*Again, this theorem falls after Theorems 17 and 18 in my deduction, so it won't be used later in their derivations.

1. $\alpha \cap \overline{\alpha} = \emptyset$	Ax. 9
2. $\overline{\overline{\alpha \cap \overline{\alpha}}} = \alpha \cap \overline{\alpha}$	Th. 9
3. $\overline{\overline{\alpha \cap \overline{\alpha}}} = \emptyset$	Substitution (1 into 2)
4. $\overline{\overline{\alpha \cap \overline{\alpha}}} = \overline{\overline{\alpha} \cup \overline{\overline{\alpha}}}$	Th. 17
5. $\overline{\overline{\alpha} \cup \overline{\overline{\alpha}}} = \emptyset$	Substitution (4 into 3)
6. $\overline{\overline{\alpha} \cup \alpha} = \emptyset$	Th. 9 (applied to $\overline{\overline{\alpha}}$)
7. $\overline{\alpha \cup \overline{\alpha}} = \emptyset$	Ax. 5
8. $\alpha \cup \overline{\alpha} = \mathbf{U}$	Ax. 9
9. $\overline{\mathbf{U}} = \emptyset$	Substitution (8 into 7)
10. $\emptyset = \overline{\mathbf{U}}$	Identity (9)

15a.) $\alpha \cup (\alpha \cap \beta) = \alpha$

*I felt it useful to prove both this theorem and its correlate (15 b), even though Copi left the latter out of his list, as they both become powerful tools in the proof of Theorem 17.

1. $\alpha \cup (\alpha \cap \beta) = (\alpha \cap \mathbf{U}) \cup (\alpha \cap \beta)$	Ax. 4
2. $(\alpha \cap \mathbf{U}) \cup (\alpha \cap \beta) = \alpha \cap (\mathbf{U} \cup \beta)$	Ax. 8
3. $\alpha \cap (\mathbf{U} \cup \beta) = \alpha \cap (\beta \cup \mathbf{U})$	Ax. 5
4. $\alpha \cap (\beta \cup \mathbf{U}) = \alpha \cap \mathbf{U}$	Th. 5
5. $\alpha \cap \mathbf{U} = \alpha$	Ax. 4
6. $\alpha \cup (\alpha \cap \beta) = \alpha$	Equivalence, 1-5

15b.) $\alpha \cap (\alpha \cup \beta) = \alpha$

1. $\alpha \cap (\alpha \cup \beta) = (\alpha \cup \emptyset) \cap (\alpha \cup \beta)$	Ax. 3
2. $(\alpha \cup \emptyset) \cap (\alpha \cup \beta) = \alpha \cup (\emptyset \cap \beta)$	Ax. 7
3. $\alpha \cup (\emptyset \cap \beta) = \alpha \cup (\beta \cap \emptyset)$	Ax. 6
4. $\alpha \cup (\beta \cap \emptyset) = \alpha \cup \emptyset$	Th. 6
5. $\alpha \cup \emptyset = \alpha$	Ax. 3
6. $\alpha \cap (\alpha \cup \beta) = \alpha$	Equivalence, 1-5

16.) $\alpha \neq \overline{\alpha}$

1. $\alpha = \overline{\alpha}$	Assume
2. $\alpha \cup \overline{\alpha} = \mathbf{U}$	Ax. 9
3. $\alpha \cup \alpha = \mathbf{U}$	Substitution (1 into 2)
4. $\alpha = \mathbf{U}$	Th. 3
5. $\alpha \cap \overline{\alpha} = \emptyset$	Ax. 9
6. $\overline{\alpha} \cap \overline{\alpha} = \emptyset$	Substitution (1 into 5)
7. $\overline{\alpha} = \emptyset$	Th. 4
8. $\mathbf{U} \neq \emptyset$	Th. 7
9. $\alpha \neq \emptyset$	Substitution (4 into 8)
10. $\alpha \neq \overline{\alpha}$	Substitution (7 into 9)
11. $(\alpha = \overline{\alpha}) \rightarrow (\alpha \neq \overline{\alpha})$	1-10 CP
12. $\sim (\alpha = \overline{\alpha}) \vee (\alpha \neq \overline{\alpha})$	11 MI
13. $(\alpha \neq \overline{\alpha}) \vee (\alpha \neq \overline{\alpha})$	Equivalence, 12 $[\sim (\alpha = \alpha) \equiv (\alpha \neq \alpha)]$
14. $\alpha \neq \overline{\alpha}$	13, Tautology

17.) $\overline{\alpha \cap \beta} = \overline{\alpha} \cup \overline{\beta}$

*The proof for this theorem is a little complicated compared to the others in this section. First, we show that the entity $\overline{\alpha}$ in Ax. 9 is unique. This allows us to say that if $\alpha \cup \beta = \mathbf{U}$ and if $\alpha \cap \beta = \emptyset$, then $\beta = \overline{\alpha}$. Then we show that $(\alpha \cap \beta) \cup (\overline{\alpha} \cup \overline{\beta}) = \mathbf{U}$ and $(\alpha \cap \beta) \cap (\overline{\alpha} \cup \overline{\beta}) = \emptyset$. At which point we can validly deduce that $\overline{\alpha} \cup \overline{\beta} = \overline{\alpha \cap \beta}$.

Lemma 1: $\overline{\alpha}$ in Ax. 9 is unique

Suppose that there are two entities $\overline{\alpha_1}$ and $\overline{\alpha_2}$ such that:

$$\alpha \cup \overline{\alpha_1} = \alpha \cup \overline{\alpha_2} = \mathbf{U} \text{ and } \alpha \cap \overline{\alpha_1} = \alpha \cap \overline{\alpha_2} = \emptyset$$

1. $\overline{\alpha_1} = \overline{\alpha_1} \cap \mathbf{U}$	Ax. 4
2. $\overline{\alpha_1} \cap \mathbf{U} = \overline{\alpha_1} \cap (\alpha \cup \overline{\alpha_2})$	Substitution (hypothesis $\alpha \cup \overline{\alpha_2} = \mathbf{U}$)
3. $\overline{\alpha_1} \cap (\alpha \cup \overline{\alpha_2}) = (\overline{\alpha_1} \cap \alpha) \cup (\overline{\alpha_1} \cap \overline{\alpha_2})$	Ax. 8
4. $(\overline{\alpha_1} \cap \alpha) \cup (\overline{\alpha_1} \cap \overline{\alpha_2}) = (\alpha \cap \overline{\alpha_1}) \cup (\overline{\alpha_1} \cap \overline{\alpha_2})$	Ax. 6 (applied to $\overline{\alpha_1} \cap \alpha$)
5. $(\alpha \cap \overline{\alpha_1}) \cup (\overline{\alpha_1} \cap \overline{\alpha_2}) = (\alpha \cap \overline{\alpha_2}) \cup (\overline{\alpha_1} \cap \overline{\alpha_2})$	Substitution (hypothesis $\alpha \cap \overline{\alpha_1} = \alpha \cap \overline{\alpha_2}$)
6. $(\alpha \cap \overline{\alpha_2}) \cup (\overline{\alpha_1} \cap \overline{\alpha_2}) = (\overline{\alpha_2} \cap \alpha) \cup (\overline{\alpha_2} \cap \overline{\alpha_1})$	Ax. 6 (applied twice)
7. $(\overline{\alpha_2} \cap \alpha) \cup (\overline{\alpha_2} \cap \overline{\alpha_1}) = \overline{\alpha_2} \cap (\alpha \cup \overline{\alpha_1})$	Ax. 8
9. $\overline{\alpha_2} \cap (\alpha \cup \overline{\alpha_1}) = \overline{\alpha_2} \cap \mathbf{U}$	Substitution (hypothesis $\alpha \cup \overline{\alpha_1} = \mathbf{U}$)
9. $\overline{\alpha_2} \cap \mathbf{U} = \overline{\alpha_2}$	Ax. 4
10. $\overline{\alpha_1} = \overline{\alpha_2}$	Equivalence, 1-9

Lemma 2: $\alpha \cup (\bar{\alpha} \cup \beta) = \mathbf{U}$

*I had to stray from my normal method of proof for the next two lemmas, as the lines simply became too long to fit across the page with justifications alongside of them. So, I placed the justification under the equal sign between each successive step.

$\alpha \cup (\bar{\alpha} \cup \beta) = [\alpha \cup (\bar{\alpha} \cup \beta)] \cap \mathbf{U} = [\alpha \cup (\bar{\alpha} \cup \beta)] \cap (\alpha \cup \bar{\alpha})$

Ax. 4 Ax. 9 (app to **U**)

$=\{[\alpha \cup (\bar{\alpha} \cup \beta)] \cap \alpha\} \cup \{[\alpha \cup (\bar{\alpha} \cup \beta)] \cap \bar{\alpha}\} = \{\alpha \cap [\alpha \cup (\bar{\alpha} \cup \beta)]\} \cup \{\bar{\alpha} \cap [\alpha \cup (\bar{\alpha} \cup \beta)]\}$

Ax. 8 Ax. 6 (app twice, once to each side of the "conjunct")

$=\{[\alpha \cap \alpha] \cup [\alpha \cap (\bar{\alpha} \cup \beta)]\} \cup \{[\bar{\alpha} \cap \alpha] \cup [\bar{\alpha} \cap (\bar{\alpha} \cup \beta)]\} = \{\alpha \cup [\alpha \cap (\bar{\alpha} \cup \beta)]\} \cup \{[\bar{\alpha} \cap \alpha] \cup [\bar{\alpha} \cap (\bar{\alpha} \cup \beta)]\}$

Ax. 8 (app twice) Th. 4 (app to $\alpha \cap \alpha$)

$=\{\alpha \cup [\alpha \cap (\bar{\alpha} \cup \beta)]\} \cup \{[\alpha \cap \bar{\alpha}] \cup [\bar{\alpha} \cap (\bar{\alpha} \cup \beta)]\} = \{\alpha \cup [\alpha \cap (\bar{\alpha} \cup \beta)]\} \cup \{\emptyset \cup [\bar{\alpha} \cap (\bar{\alpha} \cup \beta)]\}$

Ax. 6 (app to $\bar{\alpha} \cap \alpha$) Ax. 9

$=\{\alpha \cup [\alpha \cap (\bar{\alpha} \cup \beta)]\} \cup \{[\bar{\alpha} \cap (\bar{\alpha} \cup \beta)] \cup \emptyset\} = \{\alpha \cup [\alpha \cap (\bar{\alpha} \cup \beta)]\} \cup \{\bar{\alpha} \cap (\bar{\alpha} \cup \beta)\}$

Ax. 5 Ax. 3

$=\alpha \cup \{\bar{\alpha} \cap (\bar{\alpha} \cup \beta)\} = \alpha \cup \bar{\alpha} = \mathbf{U}$

Th. 15a Th. 15b Ax. 9

Thus, $\alpha \cup (\bar{\alpha} \cup \beta) = \mathbf{U}$

Lemma 3: $\alpha \cap (\bar{\alpha} \cap \beta) = \emptyset$

$\alpha \cap (\bar{\alpha} \cap \beta) = [\alpha \cap (\bar{\alpha} \cap \beta)] \cup \emptyset = [\alpha \cap (\bar{\alpha} \cap \beta)] \cup (\alpha \cap \bar{\alpha})$

Ax. 3 Ax. 9 (app to Ø)

$=\{[\alpha \cap (\bar{\alpha} \cap \beta)] \cup \alpha\} \cap \{[\alpha \cap (\bar{\alpha} \cap \beta)] \cup \bar{\alpha}\} = \{\alpha \cup [\alpha \cap (\bar{\alpha} \cap \beta)]\} \cap \{\bar{\alpha} \cup [\alpha \cap (\bar{\alpha} \cap \beta)]\}$

Ax. 7 Ax. 5 (app twice, once to each side of the "disjunct")

$=\{[\alpha \cup \alpha] \cap [\alpha \cup (\bar{\alpha} \cap \beta)]\} \cap \{[\bar{\alpha} \cup \alpha] \cap [\bar{\alpha} \cup (\bar{\alpha} \cap \beta)]\} = \{\alpha \cap [\alpha \cup (\bar{\alpha} \cap \beta)]\} \cap \{[\bar{\alpha} \cup \alpha] \cap [\bar{\alpha} \cup (\bar{\alpha} \cap \beta)]\}$

Ax. 7 (app twice) Th. 3 (app to $\alpha \cup \alpha$)

$=\{\alpha \cap [\alpha \cup (\bar{\alpha} \cap \beta)]\} \cap \{[\alpha \cup \bar{\alpha}] \cap [\bar{\alpha} \cup (\bar{\alpha} \cap \beta)]\} = \{\alpha \cap [\alpha \cup (\bar{\alpha} \cap \beta)]\} \cap \{\mathbf{U} \cap [\bar{\alpha} \cup (\bar{\alpha} \cap \beta)]\}$

Ax. 5 (app to $\bar{\alpha} \cup \alpha$) Ax. 9

$=\{\alpha \cap [\alpha \cup (\bar{\alpha} \cap \beta)]\} \cap \{[\bar{\alpha} \cup (\bar{\alpha} \cap \beta)] \cap \mathbf{U}\} = \{\alpha \cap [\alpha \cup (\bar{\alpha} \cap \beta)]\} \cap \{\bar{\alpha} \cup (\bar{\alpha} \cap \beta)\}$

Ax. 6 Ax. 4

$=\alpha \cap \{\bar{\alpha} \cup (\bar{\alpha} \cap \beta)\} = \alpha \cap \bar{\alpha} = \emptyset$

Th. 15b Th. 15a Ax. 9

Thus, $\alpha \cap (\bar{\alpha} \cap \beta) = \emptyset$

Lemma 4: $(\alpha \cap \beta) \cup (\bar{\alpha} \cup \bar{\beta}) = \mathbf{U}$

1. $(\alpha \cap \beta) \cup (\bar{\alpha} \cup \bar{\beta}) = (\bar{\alpha} \cup \bar{\beta}) \cup (\alpha \cap \beta)$	Ax. 5
2. $(\bar{\alpha} \cup \bar{\beta}) \cup (\alpha \cap \beta) = [(\bar{\alpha} \cup \bar{\beta}) \cup \alpha] \cap [(\bar{\alpha} \cup \bar{\beta}) \cup \beta]$	Ax. 7
3. $[(\bar{\alpha} \cup \bar{\beta}) \cup \alpha] \cap [(\bar{\alpha} \cup \bar{\beta}) \cup \beta] = [\alpha \cup (\bar{\alpha} \cup \bar{\beta})] \cap [\beta \cup (\bar{\alpha} \cup \bar{\beta})]$	Ax. 5 (app twice)
4. $[\alpha \cup (\bar{\alpha} \cup \bar{\beta})] \cap [\beta \cup (\bar{\alpha} \cup \bar{\beta})] = [\alpha \cup (\bar{\alpha} \cup \bar{\beta})] \cap [\beta \cup (\bar{\beta} \cup \bar{\alpha})]$	Ax. 5 [app to 2nd $(\bar{\alpha} \cup \bar{\beta})$]
5. $[\alpha \cup (\bar{\alpha} \cup \bar{\beta})] \cap [\beta \cup (\bar{\beta} \cup \bar{\alpha})] = \mathbf{U} \cap \mathbf{U}$	Lemma 2 (app twice)
6. $\mathbf{U} \cap \mathbf{U} = \mathbf{U}$	Ax. 4
7. $(\alpha \cap \beta) \cup (\bar{\alpha} \cup \bar{\beta}) = \mathbf{U}$	Equivalence, 1-6

Lemma 5: $(\alpha \cap \beta) \cap (\bar{\alpha} \cup \bar{\beta}) = \varnothing$

1. $(\alpha \cap \beta) \cap (\bar{\alpha} \cup \bar{\beta}) = [(\alpha \cap \beta) \cap \bar{\alpha}] \cup [(\alpha \cap \beta) \cap \bar{\beta}]$	Ax. 8
2. $[(\alpha \cap \beta) \cap \bar{\alpha}] \cup [(\alpha \cap \beta) \cap \bar{\beta}] = [\bar{\alpha} \cap (\alpha \cap \beta)] \cup [\bar{\beta} \cap (\alpha \cap \beta)]$	Ax. 6 (app twice)
3. $[\bar{\alpha} \cap (\alpha \cap \beta)] \cup [\bar{\beta} \cap (\alpha \cap \beta)] = [\bar{\alpha} \cap (\alpha \cap \beta)] \cup [\bar{\beta} \cap (\beta \cap \alpha)]$	Ax. 6 [app to 2nd $(\alpha \cap \beta)$]
4. $[\bar{\alpha} \cap (\alpha \cap \beta)] \cup [\bar{\beta} \cap (\beta \cap \alpha)] = [\bar{\alpha} \cap (\bar{\bar{\alpha}} \cap \beta)] \cup [\bar{\beta} \cap (\bar{\bar{\beta}} \cap \alpha)]$	Th. 9 (app twice)
5. $[\bar{\alpha} \cap (\bar{\bar{\alpha}} \cap \beta)] \cup [\bar{\beta} \cap (\bar{\bar{\beta}} \cap \alpha)] = \varnothing \cup \varnothing$	Lemma 3 (app twice)
6. $\varnothing \cup \varnothing = \varnothing$	Ax. 3
7. $(\alpha \cap \beta) \cap (\bar{\alpha} \cup \bar{\beta}) = \varnothing$	Equivalence, 1-6

By the uniqueness of $\bar{\alpha}$ in Ax. 9, provided by Lemma 1, and the fact that $(\alpha \cap \beta) \cup (\bar{\alpha} \cup \bar{\beta}) = \mathbf{U}$ and $(\alpha \cap \beta) \cap (\bar{\alpha} \cup \bar{\beta}) = \varnothing$, provided by Lemmas 4 and 5, we have the desired result:

$$\bar{\alpha} \cup \bar{\beta} = \overline{\alpha \cap \beta}$$

18.) $\overline{\alpha \cup \beta} = \bar{\alpha} \cap \bar{\beta}$

1. $\alpha \cup \beta = \bar{\bar{\alpha}} \cup \bar{\bar{\beta}}$	Ax. 9 (applied twice)
2. $\bar{\bar{\alpha}} \cup \bar{\bar{\beta}} = \overline{\bar{\alpha} \cap \bar{\beta}}$	Th. 17
3. $\alpha \cup \beta = \overline{\bar{\alpha} \cap \bar{\beta}}$	Substitution (2 into 1)
4. $\bar{\alpha} \cap \bar{\beta} = \overline{\alpha \cup \beta}$	Th. 8

19.) If $\alpha \cap \overline{\beta} = \emptyset$ and $\beta \cap \overline{\gamma} = \emptyset$, then $\alpha \cap \overline{\gamma} = \emptyset$.

1. $[\alpha \cap \overline{\beta} = \emptyset] \bullet [\beta \cap \overline{\gamma} = \emptyset]$	Assume
2. $\alpha \cap \overline{\beta} = \emptyset$	1 Simp
3. $\beta \cap \overline{\gamma} = \emptyset$	2 Simp
4. $\alpha \cap \overline{\gamma} = (\alpha \cap \mathbf{U}) \cap \overline{\gamma}$	Substitution (Ax. 4)
5. $(\alpha \cap \mathbf{U}) \cap \overline{\gamma} = [\alpha \cap (\beta \cup \overline{\beta})] \cap \overline{\gamma}$	Substitution (Ax. 9)
6. $[\alpha \cap (\beta \cup \overline{\beta})] \cap \overline{\gamma} = \overline{\gamma} \cap [\alpha \cap (\beta \cup \overline{\beta})]$	Ax. 6
7. $\overline{\gamma} \cap [\alpha \cap (\beta \cup \overline{\beta})] = \overline{\gamma} \cap [(\alpha \cap \beta) \cup (\alpha \cap \overline{\beta})]$	Ax. 8 (applied to inside [ ])
8. $\overline{\gamma} \cap [(\alpha \cap \beta) \cup (\alpha \cap \overline{\beta})] = [\overline{\gamma} \cap (\alpha \cap \beta)] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})]$	Ax. 8
9. $[\overline{\gamma} \cap (\alpha \cap \beta)] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})] = [(\alpha \cap \beta) \cap \overline{\gamma}] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})]$	Ax. 6
10. $[(\alpha \cap \beta) \cap \overline{\gamma}] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})] = [\alpha \cap (\beta \cap \overline{\gamma})] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})]$	Th. 13
11. $[\alpha \cap (\beta \cap \overline{\gamma})] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})] = [\alpha \cap \emptyset] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})]$	Substitution (3)
12. $[\alpha \cap \emptyset] \cup [\overline{\gamma} \cap (\alpha \cap \overline{\beta})] = [\alpha \cap \emptyset] \cup [\overline{\gamma} \cap \emptyset]$	Substitution (2)
13. $[\alpha \cap \emptyset] \cup [\overline{\gamma} \cap \emptyset] = \emptyset \cap \emptyset$	Th. 6 (applied twice)
14. $\emptyset \cap \emptyset = \emptyset$	Th. 6
15. $\alpha \cap \overline{\gamma} = \emptyset$	Equivalence, 4-14
16. $\{[\alpha \cap \overline{\beta} = \emptyset] \bullet [\beta \cap \overline{\gamma} = \emptyset]\} \rightarrow \{\alpha \cap \overline{\gamma} = \emptyset\}$	1-15, CP