8.2 Primitive Symbols and Well Formed Formulas

Exercises on pg. 217-218

1.) $(\sim(A_1))\bullet(A_1)$ is a wff, by (c), if $\sim(A_1)$ and A_1 are wffs
$\sim(A_1)$ is a wff, by (b), if A_1 is a wff
A_1 is a wff, by (a)
Thus, $(\sim(A_1))\bullet(A_1)$ is a wff

2.) $\sim(\sim((B_1)\bullet(\sim(C_3))))$ is a wff, by (b), if $\sim((B_1)\bullet(\sim(C_3)))$ is a wff
$\sim((B_1)\bullet(\sim(C_3)))$ is a wff, by (b), if $(B_1)\bullet(\sim(C_3))$ is a wff
$(B_1)\bullet(\sim(C_3))$ is a wff, by (b), if B_1 and $\sim(C_3)$ are wffs
$\sim(C_3)$ is a wff, by (b), if C_3 is a wff
C_3 is a wff, by (a)
B_1 is a wff, by (a)
Thus, $\sim(\sim((B_1)\bullet(\sim(C_3))))$ is a wff

3.) $\sim(\sim((B_2)\bullet(\sim(D_4)))$ is a wff, by (b), if $\sim((B_2)\bullet(\sim(D_4))$ is a wff
$\sim((B_2)\bullet(\sim(D_4))$ is a wff, by (b), if $(B_2)\bullet(\sim(D_4)$ is a wff
$(B_2)\bullet(\sim(D_4)$ is a wff, by (c), if B_2 and $\sim(D_4$ is a wff
$\sim(D_4$ is not a wff
Thus, $\sim(\sim((B_2)\bullet(\sim(D_4))$ is not a wff

4.) $\sim((\sim(D_3))\bullet(\sim(D_4)))$ is a wff, by (b), if $(\sim(D_3))\bullet(\sim(D_4))$ is a wff
$(\sim(D_3))\bullet(\sim(D_4))$ is a wff, by (c), if $\sim(D_3)$ and $\sim(D_4)$ are wffs
$\sim(D_3)$ and $\sim(D_4)$ are wffs, by (b) x 2, if D_3 and D_4 are wffs
D_3 and D_4 are wffs, by (a) x 2
Thus, $\sim((\sim(D_3))\bullet(\sim(D_4)))$ is a wff

5.) $(\sim((\sim)\bullet(C_2)))\bullet(\sim((B_3)\bullet(B_3)))$ is a wff, by (c), if $\sim((\sim)\bullet(C_2))$ and $\sim((B_3)\bullet(B_3))$ are wffs
$\sim((\sim)\bullet(C_2))$ is a wff, by (b), if $(\sim)\bullet(C_2)$ is a wff
$(\sim)\bullet(C_2)$ is a wff, by (c), if $\sim$ and C_2 are wffs
$\sim$ is not a wff
Thus, $(\sim((\sim)\bullet(C_2)))\bullet(\sim((B_3)\bullet(B_3)))$ is not a wff

6.) $\sim(((A)\bullet(B))\bullet(\sim(B)))$ is a wff, by (b), if $((A)\bullet(B))\bullet(\sim(B))$ is a wff
$((A)\bullet(B))\bullet(\sim(B))$ is a wff, by (c), if $(A)\bullet(B)$ and $\sim(B)$ are wffs
$(A)\bullet(B)$ is a wff, by (c), if A and B are wffs
A and B are wffs, by (a) x 2
$\sim(B)$ is a wff, by (b), since B is a wff
Thus, $\sim(((A)\bullet(B))\bullet(\sim(B)))$ is a wff

7.) $\sim((A)\bullet(B))\bullet(\sim((A)\bullet(A))))$ is a wff, by (b), if $(A)\bullet(B))\bullet(\sim((A)\bullet(A)))$ is a wff
$(A)\bullet(B))\bullet(\sim((A)\bullet(A)))$ is a wff, by (c), if $A)\bullet(B)$ and $\sim((A)\bullet(A))$ are wffs
$A)\bullet(B)$ is not a wff
Thus, $\sim((A)\bullet(B))\bullet(\sim((A)\bullet(A))))$ is not a wff

8.) (~(~(A))•(B)•(C))))•(((~(A))•(B))•(~(C))) is a wff, by (c),
if ~(~(A))•(B)•(C))) and ((~(A))•(B))•(~(C)) are wffs
~(~(A))•(B)•(C))) is a wff, by (b), if ~(A))•(B)•(C)) is a wff
~(A))•(B)•(C)) is a wff, by (b), if A))•(B)•(C) is a wff
A))•(B)•(C) is not a wff
Thus, (~(~(A))•(B)•(C))))•(((~(A))•(B))•(~(C))) is not a wff

9.) ~((~((A)•(B))))•(~((~((B)•(C)))•(~(~((C)•(A)))))) is a wff, by (b),
if (~((A)•(B))))•(~((~((B)•(C)))•(~(~((C)•(A))))) is a wff
(~((A)•(B))))•(~((~((B)•(C)))•(~(~((C)•(A))))) is a wff, by (c),
if ~((A)•(B))))•(~((~((B)•(C)) and ~(~((C)•(A)))) are wffs
~(~((C)•(A)))) is a wff, by (b), if ~((C)•(A))) is a wff
~((C)•(A))) is a wff, by (b), if (C)•(A)) is a wff
(C)•(A)) is a wff, by (c), if C and A) are wffs
A) is not a wff
Thus, ~((~((A)•(B))))•(~((~((B)•(C)))•(~(~((C)•(A)))))) is not a wff

10.) ~(((~((A)•(~(B))))•(A))•(~(~((A)•(~(B)))))) is a wff, by (b),
if ((~((A)•(~(B))))•(A))•(~(~((A)•(~(B))))) is a wff
((~((A)•(~(B))))•(A))•(~(~((A)•(~(B))))) is a wff, by (c),
if (~((A)•(~(B))))•(A) and ~(~((A)•(~(B)))) are wffs
(~((A)•(~(B))))•(A) is a wff, by (c), if ~((A)•(~(B))) and A are wffs
~(~((A)•(~(B)))) is a wff, by (b), if ~((A)•(~(B))) is a wff
~((A)•(~(B))) is a wff, by (b), if (A)•(~(B)) is a wff
(A)•(~(B)) is a wff, by (c), if A and ~(B) are wffs
~(B) is a wff, by (b), if B is a wff
A and B are wffs, by (a) x 2
Thus, ~(((~((A)•(~(B))))•(A))•(~(~((A)•(~(B)))))) is a wff

Exercises on pg. 220

1.) P ∨ ~P
~((~(P))•(~(~P)))
~((~(P))•(~(~(P))))

2.) ~~P → P
(~~P) → P
~(~~P)•(~(P)))
~((~(~(P)))•(~(P)))

3.) PQ → P
(P•Q) → P
~((P•Q)•(~(P)))

4.) PQ ∨ R
(P•Q) ∨ R
~((P•Q))•(~(R)))

5.) P → (Q → P)
P → (~((Q)•(~(P))))
~((P)•(~(~((Q)•(~(P))))))

6.) P → Q → P
(P → Q) → P
~((P)•(~(Q))) → P
~((~((P)•(~(Q))))•(~(P)))

7.) P → (P ∨ Q)
P → (~((~(P))•(~(Q))))
~((P)•(~((~((~(P))•(~(Q)))))))

8.) (P → PQ) → (P → Q)
(P → P•Q) → (P → Q)
(~((P)•(~(P•Q)))) → (~((P)•(~(Q))))
~(((~((P)•(~(P•Q)))))•(~((~((P)•(~(Q)))))))

9.) (P → Q)(Q → R) → (P → R)
(P → Q)•(Q → R) → (P → R)
((~((P)•(~(Q))))•(~((Q)•(~(R))))) → (~((P)•(~(R))))
~((((~((P)•(~(Q))))•(~((Q)•(~(R))))))•(~((~((P)•(~(R)))))))

10.) P → ~P ≡ ~P
(P → (~(P))) ≡ (~(P))
(~((P)•(~((~(P)))))) ≡ (~(P))
(~(((~((P)•(~((~(P)))))))•(~((~(P))))))•(~(((~(P)))•(~((~((P)•(~((~(P))))))))))

Exercises on pg. 222

1.) f_1 (P,Q) can be expressed as: ~(P•Q)
f_2 (P,Q) can be expressed as: ~(P•~Q)
f_3 (P,Q) can be expressed as: ~(~P•Q)
f_4 (P,Q) can be expressed as: ~(~P•~Q)
f_5 (P,Q) can be expressed as: ~(P•Q)•~(P•~Q)
f_6 (P,Q) can be expressed as: ~(P•Q)•~(~P•Q)
f_7 (P,Q) can be expressed as: ~(P•Q)•~(~P•~Q)
f_8 (P,Q) can be expressed as: ~(P•~Q)•~(~P•Q)
f_9 (P,Q) can be expressed as: ~(P•~Q)•~(~P•~Q)
f_{10} (P,Q) can be expressed as: ~(~P•Q)•~(~P•~Q)
f_{11} (P,Q) can be expressed as: ~P•~Q
f_{12} (P,Q) can be expressed as: ~P•Q
f_{13} (P,Q) can be expressed as: P•~Q
f_{14} (P,Q) can be expressed as: P•Q
f_{15} (P,Q) can be expressed as: (P•~P)•Q
f_{16} (P,Q) can be expressed as: ~((P•~P)•Q)

2.)

P	Q	R	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
T	T	T	F	T	T	T	T	T	T	T	F	F
T	T	F	T	F	T	T	T	T	T	T	F	T
T	F	T	T	T	F	T	T	T	T	T	T	F
T	F	F	T	T	T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	F	T	T	T	T	T
F	T	F	T	T	T	T	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T	F	T	T	T
F	F	F	T	T	T	T	T	T	T	F	T	T

f_1: ~(P•Q•R)
f_2: ~(P•Q•~R)
f_3: ~(P•~Q•R)
f_4: ~(P•~Q•~R)
f_5: ~(~P•Q•R)
f_6: ~(~P•Q•~R)
f_7: ~(~P•~Q•R)
f_8: ~(~P•~Q•~R)
f_9: ~(P•Q•R)•~(P•Q•~R)
f_{10}: ~(P•Q•R)•~(P•~Q•R)

Exercises on pg. 226-227

3.) → and ~

S(→,~) is functionally complete because it contains the same '~' as R.S., and the '•' of R.S. is definable in S(→,~) as P•Q = df ~(P → ~Q). Thus, any truth function expressible in R.S. is expressible in S(→,~).

4.) → and ∨

S(→,∨) is functionally incomplete because it lacks a wff which can express a truth function whose value is false when all its arguments have the value true. Proof follows by strong induction on the number of symbols in the wff g(P,Q,R,...) of S(→,∨):

α) If g(P,Q,R,...) contains just one symbol, it is either P or Q or R or … If these are all true, then g(P,Q,R,...) cannot have the value false.

β) Assume that any wff containing < m symbols cannot be false when all its arguments are true. Let g(P,Q,R,...) be a wff containing exactly m symbols, with m > 1. It must be either g_1(P,Q,R,...) → g_2(P,Q,R,...) or g_1(P,Q,R,...) ∨ g_2(P,Q,R,...), where g_1(P,Q,R,...) and g_2(P,Q,R,...) are wffs of S(→,∨) containing < m connectives. Since g_1(P,Q,R,...) and g_2(P,Q,R,...) each contain < m symbols, they must have the value true if all their arguments are true (by the induction hypothesis). And since T → T and T ∨ T both result in values of T, g(P,Q,R,...) cannot be false when all of its arguments are true. Thus, no wff of S(→,∨) can be false when its arguments are all true. Ergo, S(→,∨) is functionally incomplete.

5.) ~ and + (+ is exclusive disjunction)

S(~,+) is functionally incomplete because any wff of S(~,+) constructed from P,Q,~,+ has 0, 2, or 4 T's in its 4-row truth table. Proof follows by strong induction on the number of connectives in any wff g(P,Q) of S(~,+):

α) If g(P,Q) contains no connectives, then it is either P or Q. Thus, its truth table would be:

P	Q	g(P,Q) ≡ P
T	T	T
T	F	T
F	T	F
F	F	F

or

P	Q	g(P,Q) ≡ Q
T	T	T
T	F	F
F	T	T
F	F	F

In either case, g(P,Q) has 2 T's in its truth table.

β) Assume that any wff of S(~,+) constructed from P,Q,~,+ containing < m connectives, has 0, 2, or 4 T's in its truth table. Let g(P,Q) be a wff of S(~,+) containing exactly m connectives, with m > 0. g(P,Q) must be either ~g1(P,Q) or g_1(P,Q) + g_2(P,Q), where g_1(P,Q) and g_2(P,Q) are wffs of S(~,+) with < m connectives. By the induction hypothesis, g_1(P,Q) and g_2(P,Q) have 0, 2, or 4 T's in their truth tables.

Case 1: g(P,Q) ≡ ~g_1(P,Q)

Since the '~' connective merely switches the truth values of that which it operates on, and since g_1(P,Q) has only 0, 2, or 4 T's in its truth table, ~g_1(P,Q) must have 0, 2, or 4 F's in its truth table. But this implies that it must have 4, 2, or 0 T's in its truth table.

Case 2: g(P,Q) ≡ [g_1(P,Q) + g_2(P,Q)]

Here there are a number of sub-cases [In this section let 'A' stand for 'g_1(P,Q)' and let 'B' stand for 'g_2(P,Q)']: If A and B each have 0 T's, then A + B will have 0 T's. If A has 0 T's and B has 2 T's, then A + B will have 2 T's. If A has 0 T's and B has 4 T's, then A + B will have 4 T's. If A has 4 T's and B has 0 T's, then A + B will have 4 T's. If A has 4 T's and B has 2 T's, then A + B will have 2 T's. If A and B each have 4 T's, then A + B will have 0 T's. If A has 2 T's and B has 0 T's, then A + B will have 2 T's. If A has 2 T's and B has 4 T's, then A + B will have 2 T's. And, if A and B each have 2 T's, then A + B will depend on the arrangement of the truth values (since A + B is logically equivalent to B + A, we can omit any symmetrical redundancies):

A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
T	T	T	T	T	T	T	F	T	F	T	F	T	T	T	T	T	F	T	F	T	F	T	T	T	F	T	F	T	F	F	F	F	F	F	F	F	F	F	F	F	F
T	T	T	F	T	F	T	T	T	T	T	F	F	F	F	F	F	T	F	T	F	F	F	F	F	T	F	T	F	F	T	T	T	T	T	F	T	T	T	F	F	F
F	F	F	T	F	F	F	T	F	F	F	T	T	T	T	F	T	T	T	F	T	T	F	F	F	T	F	F	F	T	T	T	T	F	T	T	F	F	F	T	T	T
F	F	F	F	F	T	F	F	F	T	F	T	F	F	F	T	F	F	F	T	F	T	T	T	T	F	T	T	T	T	F	F	F	T	F	T	T	T	T	T	T	T

In all 21 possible combinations of truth values, A + B will have 0, 2, or 4 T's in its truth table. Thus, we can infer by induction that every wff of S(~,+) constructed from P,Q,~,+ has 0, 2, or 4 T's in its truth table. This implies that no wff of S(~,+), so constructed, has 1 or 3 T's in its truth table, leaving a number of truth functions inexpressible with the allotted set of connectives {~,+}. Thus, S(~,+) is functionally incomplete.

7.) $\vee$ and +

S($\vee$,+) is functionally incomplete because it lacks a wff which can express a truth function whose value is true when all its arguments have the value false. Proof follows by strong induction on the number of symbols in the wff g(P,Q,R,...) of S($\vee$,+):

α) If g(P,Q,R,...) contains just one symbol, it is either P or Q or R or... If these are all false, then g(P,Q,R,...) cannot have the value true.

β) Assume that any wff of S($\vee$,+) containing < m symbols cannot be true when all of its arguments are false. Let g(P,Q,R,...) be a wff containing exactly m symbols, with m > 1. g(P,Q,R,...) must be either g_1(P,Q,R,...) $\vee$ g_2(P,Q,R,...) or g_1(P,Q,R,...) + g_2(P,Q,R,...), where g_1(P,Q,R,...) and g_2(P,Q,R,...) are wffs of S($\vee$,+) containing < m symbols. By the induction hypothesis, g_1(P,Q,R,...) and g_2(P,Q,R,...) must have the value false when all of their arguments are false. Since F $\vee$ F and F + F both result in a truth value of F, we know g(P,Q,R,...) cannot have a truth value of T when the truth values of all its arguments are F. Thus, no wff of S($\vee$,+) can be true when its arguments are all false. This implies that S($\vee$,+) is functionally incomplete.

8.) + and ≡

S(+,≡) is functionally incomplete because any wff of S(+,≡) constructed from P,Q,+,≡ has 0, 2, or 4 T's in its truth table. Again, the proof follows by strong induction on the number of connectives in any wff g(P,Q) of S(+,≡):

α) The α-case is identical to that of the α-case in proof (5) above.
So, if g(P,Q) contains no connectives, then it has 2 T's in its truth table.

β) Assume that any wff of S(+,≡) constructed from P,Q,+,≡ containing < m connectives has 0, 2, or 4 T's in its truth table. Let g(P,Q) be a wff of S(+,≡) containing exactly m connectives, with m > 0. g(P,Q) must be either g_1(P,Q) + g_2(P,Q) or g_1(P,Q) ≡ g_2(P,Q), where g_1(P,Q) and g_2(P,Q) are wffs of S(+,≡) containing < m connectives. By the induction hypothesis, g_1(P,Q) and g_2(P,Q) have 0, 2, or 4 T's in their truth tables.

Case 1: g(P,Q) ≡ [g_1(P,Q) + g_2(P,Q)]

We have already seen, via exercise (5), that g_1(P,Q) + g_2(P,Q) has 0, 2, or 4 T's in its truth table.

Case 2: g(P,Q) ≡ [g_1(P,Q) ≡ g_2(P,Q)]

The '≡' connective is the opposite of the '+' connective, in the sense that, given a truth value for the truth function A + B, the truth value for the truth function A ≡ B will be its opposite:

A	B	A + B	A ≡ B
T	T	F	T
T	F	T	F
F	T	T	F
F	F	F	T

It follows then that since g_1(P,Q) + g_2(P,Q) has 0, 2, or 4 T's in its truth table, g_1(P,Q) ≡ g_2(P,Q) necessarily has 0, 2, or 4 F's in its truth table. This implies that g_1(P,Q) ≡ g_2(P,Q) has 4, 2, or 0 T's in its truth table, respectively. So, g(P,Q) must have 0, 2, or 4 T's in its truth table. Thus, we can infer by induction that any wff of S(+,≡) constructed from P,Q,+,≡ has 0, 2, or 4 T's in its truth table. This implies that no wff of S(+,≡), so constructed, has 1 or 3 T's in its truth table, leaving a number of truth functions inexpressible with the allotted set of connectives {+,≡}. Ergo, S(+,≡) is functionally incomplete.

9.) + and •

S(+,•) is functionally incomplete because it lacks a wff which can express a truth function whose value is true when all of its arguments have the value of false. (The proof is identical to exercise (7) above, with a simple swap of connectives) g(P,Q,R,...) must be either g_1(P,Q,R,...) + g_2(P,Q,R,...) or g_1(P,Q,R,...) • g_2(P,Q,R,...). Since F + F and F • F both result in a truth value of F, g(P,Q,R,...) cannot have a truth value of T when all of its arguments have F as their truth values. Thus, no wff of S(+,•) can be true when all of its arguments are false. This implies that S(+,•) is functionally incomplete.

10.) ≡ and ~

S(≡,~) is functionally incomplete for the same reason S(+,~) from exercise (5) was functionally incomplete. The proof is identical to that of (5), with a simple substitution of '≡' for '+'. The symmetry between these two connectives allows one to conclude that in each case where A + B resulted in 2 T's, A ≡ B would as well. In those cases where A + B resulted in 0 or 4 T's, A ≡ B would result in 4 or 0 T's, respectively. Thus, we can infer by induction that every wff of S(≡,~) constructed from P,Q,≡,~ has 0, 2, or 4 T's in its truth table. This implies that no wff of S(≡,~), so constructed, has 1 or 3 T's in its truth table. Thus, S(≡,~) is functionally incomplete.

11.) ≡ and →

S(≡,→) is functionally incomplete for the same reason S(→,∨) from exercise (4) was functionally incomplete. The proof is identical to that of (4), with a simple substitution of '≡' for '∨'. Thus, no wff of S(≡,→) can be false when its arguments are all true. This implies that S(≡,→) is functionally incomplete.

12.) → and ↛

S(→,↛) is functionally complete because the '•' of R.S. is definable in S(→,↛) as P•Q = df P ↛ (P ↛ Q) and the '~' of R.S. is definable in S(→,↛) as ~P = df (P → P) ↛ P. Thus, any truth function expressible in R.S. is expressible in S(→,↛).

13.) → and ↚

S(→,↚) is functionally complete because the '•' of R.S. is definable in S(→,↚) as P•Q = df (P ↚ Q) ↚ Q and the '~' of R.S. is definable in S(→,↚) as ~P = df P → (P ↚ P) Thus, any truth function expressible in R.S. is expressible in S(→,↚).

14.) ↛ and ↚

S(↛,↚) is functionally incomplete for the same reason that S(∨,+) from exercise (7) was functionally incomplete. The proof is identical to that of (7), with a substitution of connectives: '↛' for '∨' and '↚' for '+'. Thus, no wff of S(↛,↚) can be true when all of its arguments are false. This implies that S(↛,↚) is functionally incomplete.

15.) Only f_1(P,Q), which we will signify with the symbol '↑', and f_{11}(P,Q), which we will signify with the symbol '↓', can be added to the propositional symbols P,Q,R,... and parenthesis, to give a functionally complete logistic system with a single operator symbol. We begin by tackling the larger task of demonstrating the functional incompleteness of the other 14 binary truth functional operators. If S(→,∨) is functionally incomplete, as we have seen in exercise (4), then it necessarily follows that both S(→) and S(∨) are functionally incomplete as well. Utilizing this same reasoning in relation to all the proofs in this section, we can quickly conclude that →[f_2], ∨[f_4], •[f_{14}], +[f_7], ≡[f_8], ↛[f_{13}], ↚[f_{12}] all fail, by themselves, to form a functionally complete logistic system.

It remains to be seen if any of ←[f_3], [f_{10}], [f_5], [f_9], [f_6], F[f_{15}], T[f_{16}] form a functionally complete logistic system. S(←) is functionally incomplete for the same reason as S(→,∨) from exercise (4). The proof is identical to (4) with the deletion of '∨' and the substitution of '←' for '→'. Thus, no wff of S(←) can be false when all its arguments are true. S([f_{10}]), S([f_5]), S([f_9]), S([f_6]) are all functionally incomplete for the same reason. Their truth values are solely dependent upon the truth values of one of the propositional symbols/functions which they connect. They could be symbolized as P, ~P, Q, ~Q, respectively. Thus, any wff within any of these sets will necessarily have 2 T's and 2 F's in its truth table. This leaves a great number of truth functions inexpressible. So, none of these are sufficient to give functional completeness. The final two are trivial. S(F[f_{15}]) is functionally incomplete because no truth functions are expressible except that which characterizes a contradiction, namely all F's. And S(T[f_{16}]) is functionally incomplete because no truth functions are expressible except that of the tautology, or all T's.

Now that the other 14 binary truth functional operators have been shown to be functionally incomplete when they are considered individually, we can move on and prove the functional completeness of S(↑) and S(↓). S(↑) is functionally complete because the '~' of R.S. is definable in S(↑) as ~P = df P ↑ P and the '•' of R.S. is definable in S(↑) as P•Q = df (P ↑ Q) ↑ (P ↑ Q). S(↓) is functionally complete because the '~' of R.S. is definable in S(↓) as ~P = df P ↓ P and the '•' of R.S. is definable in S(↓) as P•Q = df (P ↓ P) ↓ (Q ↓ Q). Thus, any truth function expressible in R.S. is expressible in both S(↑) and S(↓). This implies that both S(↑) and S(↓) are functionally complete sets consisting of only a single operator symbol.

8.4 Independence of the Axioms

Exercises on pg. 237 (For the sake of brevity I have omitted the tables showing that the characteristic belongs to each of the axioms other than the one being shown to be independent – but these are easily filled in with the two definition tables given for each demonstration below. I have also only shown the portion of the table in which it can be seen that the characteristic does not belong, as the rest of the table is irrelevant to the demonstration)

2.) P_G System (Gotlind - Rasiowa)
Primitive Operators: ~ and v
P → Q = df ~P v Q
R1: P and P → Q, infer Q
Ax. 1: (P v P) → P
Ax. 2: P → (P v Q)
Ax. 3: (Q → R) → [(P v Q) → (R v P)]

First we show the independence of Ax. 1:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	2
1	1
2	0

P	Q	P v Q	P → Q
0	0	0	0
0	1	0	1
0	2	0	2
1	0	0	0
1	1	0	0
1	2	1	1
2	0	0	0
2	1	1	0
2	2	2	0

The characteristic belongs to both Ax. 2 and Ax. 3
However, the characteristic does not belong to Ax. 1:

(P	V	P)	→	P
1	0	1	1	1

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 1 is independent

Now we show the independence of Ax. 2:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	1
1	0
2	2

P	Q	P v Q	P → Q
0	0	0	0
0	1	0	1
0	2	0	1
1	0	0	0
1	1	1	0
1	2	1	0
2	0	0	0
2	1	1	1
2	2	1	1

The characteristic belongs to both Ax. 1 and Ax. 3
However, the characteristic does not belong to Ax. 2:

P	→	(P	v	Q)
2	1	2	1	1

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 2 is independent

Now we show the independence of Ax. 3:
We use the set {0,1,2,3} with 0 as the designated element

P	~P
0	1
1	0
2	3
3	0

P	Q	P v Q	P → Q
0	0	0	0
0	1	0	1
0	2	0	2
0	3	0	3
1	0	0	0
1	1	1	0
1	2	2	0
1	3	3	0
2	0	0	0
2	1	2	3
2	2	2	0
2	3	0	3
3	0	0	0
3	1	3	0
3	2	0	0
3	3	3	0

The characteristic belongs to both Ax. 1 and Ax. 2
However, the characteristic does not belong to Ax. 3:

(Q	→	R)	→	[(P	V	Q)	→	(R	V	P)]
3	0	1	2	2	0	3	2	1	2	2

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 3 is independent

3.) F.S. System (Frege)
Primitive Operators: ~ and →
P v Q = df ~P → Q
R1: P and P → Q, infer Q
Ax. 1: P → (Q → P)
Ax. 2: [P → (Q → R)] → [(P → Q) → (P → R)]
Ax. 3: (P → Q) → (~Q → ~P)
Ax. 4: ~~P → P
Ax. 5: P → ~~P

First we show the independence of Ax. 1:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	2
1	1
2	0

P	Q	P → Q
0	0	0
0	1	0
0	2	0
1	0	0
1	1	0
1	2	1
2	0	1
2	1	0
2	2	0

The characteristic belongs to Ax. 2, Ax. 3, Ax. 4, and Ax. 5
However, the characteristic does not belong to Ax. 1:

P	→	(Q	→	P)
2	1	0	0	2

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 1 is independent

Now we show the independence of Ax. 2:
We use the set {0,1,2,3} with 0 as the designated element

P	~P
0	1
1	0
2	2
3	3

P	Q	P → Q
0	0	0
0	1	1
0	2	0
0	3	0
1	0	0
1	1	0
1	2	2
1	3	3
2	0	0
2	1	0
2	2	0
2	3	2
3	0	0
3	1	0
3	2	0
3	3	0

The characteristic belongs to Ax. 1, Ax. 3, Ax. 4, and Ax. 5
However, the characteristic does not belong to Ax. 2:

[P	→	(Q	→	R)]	→	[(P	→	Q)	→	(P	→	R)]
0	0	2	0	1	1	0	0	2	1	0	1	1

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 2 is independent

Now we show the independence of Ax. 3:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	2
1	1
2	0

P	Q	P → Q
0	0	0
0	1	0
0	2	1
1	0	0
1	1	0
1	2	2
2	0	0
2	1	0
2	2	0

The characteristic belongs to Ax. 1, Ax. 2, Ax. 4, and Ax. 5
However, the characteristic does not belong to Ax. 3:

(P	→	Q)	→	(~	Q	→	~	P)
0	0	1	1	1	1	2	2	0

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 3 is independent

Now we show the independence of Ax. 4:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	1
1	2
2	0

P	Q	P → Q
0	0	0
0	1	0
0	2	0
1	0	0
1	1	0
1	2	2
2	0	0
2	1	0
2	2	0

The characteristic belongs to Ax. 1, Ax. 2, Ax. 3, and Ax. 5
However, the characteristic does not belong to Ax. 4:

(~	~	P)	→	P
1	0	2	2	2

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 4 is independent

Now we show the independence of Ax. 5:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	1
1	2
2	0

P	Q	P → Q
0	0	0
0	1	0
0	2	0
1	0	0
1	1	0
1	2	0
2	0	0
2	1	1
2	2	0

The characteristic belongs to Ax. 1, Ax. 2, Ax. 3, and Ax. 4
However, the characteristic does not belong to Ax. 5:

P	→	(~	~	P)
2	1	1	0	2

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 5 is independent

4.) L.S. System (Lukasiewicz)
Primitive Operators: ~ and →
P v Q = df ~P → Q
R1: P and P → Q, infer Q
Ax. 1: P → (Q → P)
Ax. 2: [P → (Q → R)] → [(P → Q) → (P → R)]
Ax. 3: (~P → ~Q) → (Q → P)

First we show the independence of Ax. 1:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	2
1	1
2	0

P	Q	P → Q
0	0	0
0	1	0
0	2	0
1	0	0
1	1	0
1	2	1
2	0	1
2	1	0
2	2	0

The characteristic belongs to both Ax. 2 and Ax. 3
However, the characteristic does not belong to Ax. 1:

P	→	(Q	→	P)
2	1	0	0	2

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 1 is independent

Now we show the independence of Ax. 2:
We use the set {0,1,2,3} with 0 as the designated element

P	~P
0	1
1	0
2	2
3	3

P	Q	P → Q
0	0	0
0	1	1
0	2	0
0	3	0
1	0	0
1	1	0
1	2	2
1	3	3
2	0	0
2	1	0
2	2	0
2	3	2
3	0	0
3	1	0
3	2	0
3	3	0

The characteristic belongs to both Ax. 1 and Ax. 3
However, the characteristic does not belong to Ax. 2:

[P	→	(Q	→	R)]	→	[(P	→	Q)	→	(P	→	R)]
0	0	2	0	1	1	0	0	2	1	0	1	1

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 2 is independent

Now we show the independence of Ax. 3:
We use the set {0,1,2} with 0 as the designated element

P	~P
0	2
1	1
2	0

P	Q	P → Q
0	0	0
0	1	2
0	2	0
1	0	0
1	1	0
1	2	0
2	0	0
2	1	1
2	2	0

The characteristic belongs to both Ax. 1 and Ax. 2
However, the characteristic does not belong to Ax. 3:

(~	P	→	~	Q)	→	(Q	→	P)
1	1	0	0	2	2	2	1	1

Finally, the characteristic 0 is hereditary with respect to R1 because the only row where both P and P → Q have the value 0, Q does as well.

Thus, Ax. 3 is independent

8.5 Development of the Calculus

Exercises on pg. 243-250

Th. 7) ⊢ P → P

1. P → (PP)	Ax. 1
2. (PP) → (PPPP)	Ax. 1
3. (PPPP) → P	Ax. 2 (P = P and Q = PPP)
4. P → P	DR. 5 (1,2,3)

Th. 8) ⊢ RP → PR

1. P → P	Th. 7
2. RP → PR	DR. 3 (Q = P)

Th. 9) ⊢ ~(PR) → ~(RP)

1. RP → PR	Th. 8
2. (RP → PR) → [~(PR) → ~(RP)]	Th. 5
3. ~(PR) → ~(RP)	R 1

DR. 7) P → Q, R → S ⊢ PR → QS

1. P → Q	Premiss
2. SP → QS	DR. 3 (R = S)
3. R → S	Premiss
4. PR → SP	DR. 3 (P = R, Q = S, R = P)
5. PR → QS	DR. 6 (4,2)

DR. 7, C1) P → Q ⊢ PR → QR

1. P → Q	Premiss
2. RP → QR	DR. 3
3. PR → RP	Th. 8
4. PR → QR	DR. 6 (3,2)

DR. 7, C2) R → S ⊢ PR → PS

1. R → S	Premiss
2. PR → SP	DR. 3
3. SP → PS	Th. 8
4. PR → PS	DR. 6 (2,3)

DR. 8) P → Q, P → R ⊢ P → (QR)

1. P → Q	Premiss
2. P → R	Premiss
3. PP → QR	DR. 7 (1,2: R = P, S = R)
4. P → PP	Ax. 1
5. P → (QR)	DR. 6 (4,3)

Th. 10, C) ⊢ P(QR) → (PQ)R

1.	P(QR) → (QR)P	Th. 8
2.	(QR)P → (QR)	Ax. 2
3.	QR → RQ	Th. 8
4.	RQ → R	Ax. 2
5.	P(QR) → RQ	DR. 5 (1,2,3)
6.	P(QR) → R	DR. 6 (5,4)
7.	P(QR) → P	Ax. 2
8.	QR → Q	Ax. 2
9.	P(QR) → Q	DR. 5 (1,2,8)
10.	P(QR) → PQ	DR. 8 (7,9)
11.	P(QR) → (PQ)R	DR. 8 (10,6)

DR. 9) P → R, Q → S ⊢ (P v Q) → (R v S)

1.	P → R	Premiss
2.	(P → R) → (~R → ~P)	Th. 5
3.	~R → ~P	R 1 (1,2)
4.	Q → S	Premiss
5.	(Q → S) → (~S → ~Q)	Th. 5
6.	~S → ~Q	R 1 (4,5)
7.	~R~S → ~P~Q	DR. 7 (3,6)
8.	[(~R~S) → (~P~Q)] → [~(~P~Q) → ~(~R~S)]	Th. 5
9.	~(~P~Q) → ~(~R~S)	R 1 (7,8)
10.	(P v Q) → ~(~R~S)	df.
11.	(P v Q) → (R v S)	df.

Th. 11) ⊢ (P v Q) → (Q v P)

1.	~Q~P → ~P~Q	Th. 8
2.	[(~Q~P) → (~P~Q)] → [~(~P~Q) → ~(~Q~P)]	Th. 5
3.	~(~P~Q) → ~(~Q~P)	R 1 (1,2)
4.	(P v Q) → ~(~Q~P)	df.
5.	(P v Q) → (Q v P)	df.

Th. 12) (P v Q) v R → P v (Q v R)

1.	~P~~(~Q~R) → (~Q~R)~P	Th. 6
2.	(~Q~R)~P → ~P(~Q~R)	Th. 8
3.	~P(~Q~R) → (~P~Q)~R	Th. 10, C
4.	~P~~(~Q~R) → (~P~Q)~R	DR. 5 (1,2,3)
5.	(~P~Q)~R → (~P~Q)	Ax. 2
6.	~P~Q → ~~(~P~Q)	Th. 4
7.	(~P~Q)~R → ~R(~P~Q)	Th. 8
8.	~R(~P~Q) → ~R	Ax. 2
9.	(~P~Q)~R → ~~(~P~Q)	DR. 5 (4,5,6)
10.	(~P~Q)~R → ~R	DR. 6 (7,8)
11.	(~P~Q)~R → ~~(~P~Q)~R	DR. 8 (9,10)
12.	~P~~(~Q~R) → ~~(~P~Q)~R	DR. 6 (4,11)
13.	[~P~~(~Q~R) → ~~(~P~Q)~R] → {~[~~(~P~Q)~R] → ~[~P~~(~Q~R)]}	Th. 5
14.	~[~~(~P~Q)~R] → ~[~P~~(~Q~R)]	R 1 (12,13)
15.	~[~(P v Q)~R] → ~[~P~(Q v R)]	df. (x2)
16.	(P v Q) v R → P v (Q v R)	df. (x2)

Th. 12, C) P v (Q v R) → (P v Q) v R

1.	~~(~P~Q)~R → ~~(~P~Q)	Ax. 2
2.	~~(~P~Q) → ~P~Q	Th. 2
3.	~P~Q → ~P	Ax. 2
4.	~~(~P~Q)~R → ~P	DR. 5 (1,2,3)
5.	~~(~P~Q)~R → ~R~~(~P~Q)	Th. 8
6.	~R~~(~P~Q) → ~R	Ax. 2
7.	~~(~P~Q)~R → ~R	DR. 6 (5,6)
8.	~~(~P~Q)~R → ~P~Q	DR. 6 (1,2)
9.	~~(~P~Q)~R → (~P~Q)~R	DR. 8 (8,7)
10.	(~P~Q)~R → ~P(~Q~R)	Th. 10
11.	~P(~Q~R) → (~Q~R)~P	Th. 8
12.	~~(~P~Q)~R → (~Q~R)~P	DR. 5 (9,10,11)
13.	(~Q~R)~P → ~Q~R	Ax. 2
14.	~Q~R → ~~(~Q~R)	Th. 4
15.	~~(~P~Q)~R → ~~(~Q~R)	DR. 5 (12,13,14)
16.	~~(~P~Q)~R → ~P~~(~Q~R)	DR. 8 (4,15)
17.	{~~(~P~Q)~R → ~P~~(~Q~R)} → {~[~P~~(~Q~R)] → ~[~~(~P~Q)~R]}	Th. 5
18.	~[~P~~(~Q~R)] → ~[~~(~P~Q)~R]	R 1 (16,17)
19.	~[~P~(Q v R)] → ~[~(P v Q)~R]	df. (x2)
20.	P v (Q v R) → (P v Q) v R	df. (x2)

Th. 13) ├ [P → (Q → R)] → [(PQ) → R]

1.	(PQ)~R → P(Q~R)	Th. 10
2.	P(Q~R) → P	Ax. 2
3.	(PQ)~R → P	DR. 6
4.	P(Q~R) → (Q~R)P	Th. 8
5.	(Q~R)P → Q~R	Ax. 2
6.	Q~R → ~~(Q~R)	Th. 4
7.	P(Q~R) → ~~(Q~R)	DR. 5 (4,5,6)
8.	(PQ)~R → ~~(Q~R)	DR. 6 (1,7)
9.	(PQ)~R → P~~(Q~R)	DR. 8 (3,8)
10.	[(PQ~R → P~~(Q~R)] → [~{P~~(Q~R)} → ~{(PQ)~R}]	Th. 5
11.	~{P~~(Q~R)} → ~{(PQ)~R}	R 1 (9,10)
12.	~{P~(Q → R)} → {(PQ) → R}	df. (x2)
13.	[P → (Q → R)] → [(PQ) → R]	df.

DR. 11) P → Q, P → (Q → R) ├ P → R

1.	P → Q	Premiss
2.	P → P	Th. 7
3.	P → PQ	DR. 8
4.	P → (Q → R)	Premiss
5.	[P → (Q → R)] → [(PQ) → R]	Th. 13
6.	(PQ) → R	R 1 (4,5)
7.	P → R	DR. 6 (3,6)

Th. 15) ├ P → (Q → PQ)

1.	PQ → PQ	Th. 7
2.	[PQ → PQ] → [P → (Q → PQ)]	Th. 14 (R = PQ)
3.	P → (Q → PQ)	R 1

Th. 16) $\vdash P \rightarrow (Q \rightarrow P)$

1. $PQ \rightarrow P$	Ax. 2
2. $[PQ \rightarrow P] \rightarrow [P \rightarrow (Q \rightarrow P)]$	Th. 14
3. $P \rightarrow (Q \rightarrow P)$	R 1

Th. 17) $\vdash P \rightarrow (Q \vee P)$

1. $\sim P \sim Q \rightarrow \sim P$	Ax. 2
2. $\sim Q \sim P \rightarrow \sim P \sim Q$	Th. 8
3. $\sim\sim(\sim Q \sim P) \rightarrow \sim Q \sim P$	Th. 2
4. $\sim\sim(\sim Q \sim P) \rightarrow \sim P$	DR. 5 (3,2,1)
5. $\sim(Q \vee P) \rightarrow \sim P$	df.
6. $[\sim(Q \vee P) \rightarrow \sim P] \rightarrow [\sim\{\sim P\} \rightarrow \sim\{\sim(Q \vee P)\}]$	Th. 5
7. $\sim\sim P \rightarrow \sim\sim(Q \vee P)$	R 1 (5,6)
8. $P \rightarrow \sim\sim P$	Th. 4
9. $\sim\sim(Q \vee P) \rightarrow (Q \vee P)$	Th. 2
10. $P \rightarrow (Q \vee P)$	Dr. 5 (8,7,9)

Th. 18') $(P \vee Q)R \quad \vdash (PR \vee QR)$

1. $(P \vee Q)R$	Premiss
2. $(P \vee Q)R \rightarrow R(P \vee Q)$	Th. 8
3. $R(P \vee Q)$	R 1 (1,2)
4. $R(P \vee Q) \rightarrow R$	Ax. 2
5. R	R 1 (3,4)
6. $R \rightarrow (P \rightarrow RP)$	Th. 15
7. $R \rightarrow (Q \rightarrow RQ)$	Th. 15
8. $P \rightarrow RP$	R 1 (5,6)
9. $Q \rightarrow RQ$	R 1 (5,7)
10. $RP \rightarrow PR$	Th. 8
11. $RQ \rightarrow QR$	Th. 8
12. $P \rightarrow PR$	DR. 6 (8,10)
13. $Q \rightarrow QR$	DR. 6 (9,11)
14. $(P \vee Q) \rightarrow (PR \vee QR)$	DR. 9 (12,13)
15. $(P \vee Q)R \rightarrow (P \vee Q)$	Ax. 2
16. $P \vee Q$	R 1 (1,15)
17. $PR \vee QR$	R 1 (16,14)

Th. 18) $\vdash (P \vee Q)R \rightarrow (PR \vee QR)$

1. $(P \vee Q)R \ \vdash (PR \vee QR)$	Th. 18'
2. $(P \vee Q)R \rightarrow (PR \vee QR)$	D.T.

DR. 12) $P \rightarrow \sim Q \ \vdash P \rightarrow \sim(QR)$

1. $P \rightarrow \sim Q$	Premiss
2. $[P \rightarrow \sim Q] \rightarrow [\sim(\sim Q) \rightarrow \sim P]$	Th. 5
3. $\sim\sim Q \rightarrow \sim P$	R 1 (1,2)
4. $Q \rightarrow \sim\sim Q$	Th. 4
5. $Q \rightarrow \sim P$	DR. 6 (4,3)
6. $QR \rightarrow Q$	Ax. 2
7. $QR \rightarrow \sim P$	Dr. 6 (6,5)
8. $[QR \rightarrow \sim P] \rightarrow [\sim(\sim P) \rightarrow \sim(QR)]$	Th. 5
9. $\sim\sim P \rightarrow \sim(QR)$	R 1 (7,8)
10. $P \rightarrow \sim\sim P$	Th. 4
11. $P \rightarrow \sim(QR)$	DR. 6 (10,9)

DR. 13) $P \rightarrow \sim R \quad \vdash P \rightarrow \sim(QR)$

1. $P \rightarrow \sim R$	Premiss
2. $P \rightarrow \sim(RQ)$	DR. 12 (Q = R, R = Q)
3. $\sim(RQ) \rightarrow \sim(QR)$	Th. 9
4. $P \rightarrow \sim(QR)$	DR. 6 (2,3)

Th. 19) $\vdash P \equiv \sim\sim P$

1. $P \rightarrow \sim\sim P$	Th. 4
2. $\sim\sim P \rightarrow P$	Th. 2
3. $(P \rightarrow \sim\sim P)(\sim\sim P \rightarrow P)$	DR. 14 (1,2)
4. $P \equiv \sim\sim P$	df.

DR. 15) $P \equiv Q \quad \vdash \sim P \equiv \sim Q$

1. $P \equiv Q$	Premiss
2. $(P \rightarrow Q)(Q \rightarrow P)$	df.
3. $(P \rightarrow Q)(Q \rightarrow P) \rightarrow (P \rightarrow Q)$	Ax. 2
4. $P \rightarrow Q$	R 1 (2,3)
5. $(P \rightarrow Q)(Q \rightarrow P) \rightarrow (Q \rightarrow P)(P \rightarrow Q)$	Th. 8
6. $(Q \rightarrow P)(P \rightarrow Q)$	R 1 (2,5)
7. $(Q \rightarrow P)(P \rightarrow Q) \rightarrow (Q \rightarrow P)$	Ax. 2
8. $Q \rightarrow P$	R 1 (6,7)
9. $(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$	Th. 5
10. $(Q \rightarrow P) \rightarrow (\sim P \rightarrow \sim Q)$	Th. 5
11. $\sim Q \rightarrow \sim P$	R 1 (4,9)
12. $\sim P \rightarrow \sim Q$	R 1 (8,10)
13. $(\sim P \rightarrow \sim Q)(\sim Q \rightarrow \sim P)$	DR. 14 (12,11)
14. $\sim P \equiv \sim Q$	df.

DR. 16) $P \equiv Q, R \equiv S \quad \vdash PR \equiv QS$

1. $P \equiv Q$	Premiss
2. $R \equiv S$	Premiss
3. $(P \rightarrow Q)(Q \rightarrow P)$	df.
4. $(R \rightarrow S)(S \rightarrow R)$	df.
5. $(P \rightarrow Q)(Q \rightarrow P) \rightarrow (P \rightarrow Q)$	Ax. 2
6. $(R \rightarrow S)(S \rightarrow R) \rightarrow (R \rightarrow S)$	Ax. 2
7. $P \rightarrow Q$	R 1 (3,5)
8. $R \rightarrow S$	R 1 (4,6)
9. $PR \rightarrow QS$	DR. 7 (7,8)
10. $(P \rightarrow Q)(Q \rightarrow P) \rightarrow (Q \rightarrow P)(P \rightarrow Q)$	Th. 8
11. $(R \rightarrow S)(S \rightarrow R) \rightarrow (S \rightarrow R)(R \rightarrow S)$	Th. 8
12. $(Q \rightarrow P)(P \rightarrow Q)$	R 1 (3,10)
13. $(S \rightarrow R)(R \rightarrow S)$	R 1 (4,11)
14. $(Q \rightarrow P)(P \rightarrow Q) \rightarrow (Q \rightarrow P)$	Ax. 2
15. $(S \rightarrow R)(R \rightarrow S) \rightarrow (S \rightarrow R)$	Ax. 2
16. $Q \rightarrow P$	R 1 (12,14)
17. $S \rightarrow R$	R 1 (13,15)
18. $QS \rightarrow PR$	DR. 7 (16,17)
19. $(PR \rightarrow QS)(QS \rightarrow PR)$	DR. 14 (9,18)
20. $PR \equiv QS$	df.

DR. 16, C) P ≡ Q, R ≡ S ├ P v R ≡ Q v S

1. P ≡ Q	Premiss
2. R ≡ S	Premiss
3. (P → Q)(Q → P)	df.
4. (R → S)(S → R)	df.
5. (P → Q)(Q → P) → (P → Q)	Ax. 2
6. (R → S)(S → R) → (R → S)	Ax. 2
7. P → Q	R 1 (3,5)
8. R → S	R 1 (4,6)
9. P v R → Q v S	DR. 9 (7,8)
10. (P → Q)(Q → P) → (Q → P)(P → Q)	Th. 8
11. (R → S)(S → R) → (S → R)(R → S)	Th. 8
12. (Q → P)(P → Q)	R 1 (3,10)
13. (S → R)(R → S)	R 1 (4,11)
14. (Q → P)(P → Q) → (Q → P)	Ax. 2
15. (S → R)(R → S) → (S → R)	Ax. 2
16. Q → P	R 1 (12,14)
17. S → R	R 1 (13,15)
18. Q v S → P v R	DR. 9 (16,17)
19. [(P v R) → (Q v S)][(Q v S) → (P v R)]	DR. 14 (9,18)
20. P v R ≡ Q v S	df.

DR. 17) P → Q, ~Q ├ ~P

1. P → Q	Premiss
2. ~Q	Premiss
3. (P → Q) → (~Q → ~P)	Th. 5
4. ~Q → ~P	R 1 (1,3)
5. ~P	R 1 (2,4)

DR. 18) P v Q, ~P ├ Q

1. P v Q	Premiss
2. ~P	Premiss
3. ~(~P~Q)	df. (1)
4. ~P → Q	df. (3)
5. Q	R 1 (2,4)

DR. 19) PQ ├ P

1. PQ	Premiss
2. PQ → P	Ax. 2
3. P	R 1 (1,2)

DR. 19, C) PQ ├ Q

1. PQ	Premiss
2. PQ → QP	Th. 8
3. QP → Q	Ax. 2
4. PQ → Q	DR. 6 (2,3)
5. Q	R 1 (1,4)

DR. 21) $(P \to Q)(R \to S), \sim Q \vee \sim S \vdash \sim P \vee \sim R$

1. $(P \to Q)(R \to S)$	Premiss
2. $P \to Q$	DR. 19 (1)
3. $R \to S$	DR. 19, C (1)
4. $(P \to Q) \to (\sim Q \to \sim P)$	Th. 5
5. $(R \to S) \to (\sim S \to \sim R)$	Th. 5
6. $\sim Q \to \sim P$	R 1 (2,4)
7. $\sim S \to \sim R$	R 1 (3,5)
8. $(\sim Q \vee \sim S) \to (\sim P \vee \sim R)$	DR. 9 (6,7)
9. $\sim Q \vee \sim S$	Premiss
10. $\sim P \vee \sim R$	R 1 (9,8)

DR. 22) $P \vdash P \vee Q$

1. P	Premiss
2. $P \to P \vee Q$	Th. 17, C
3. $P \vee Q$	R 1 (1,2)

Th. 20) $\vdash P \vee Q \equiv Q \vee P$

1. $P \to Q \vee P$	Th. 17
2. $Q \to Q \vee P$	Th. 17, C
3. $P \vee Q \to Q \vee P$	DR. 9 (1,2)
4. $Q \to P \vee Q$	Th. 17
5. $P \to P \vee Q$	Th. 17, C
6. $Q \vee P \to P \vee Q$	DR. 9 (4,5)
7. $[(P \vee Q) \to (Q \vee P)][(Q \vee P) \to (P \vee Q)]$	DR. 14 (3,6)
8. $P \vee Q \equiv Q \vee P$	df.

Th. 21) $\vdash PQ \equiv QP$

1. $PQ \to QP$	Th. 8
2. $QP \to PQ$	Th. 8
3. $(PQ \to QP)(QP \to PQ)$	DR. 14 (1,2)
4. $PQ \equiv QP$	df.

Th. 22) $[P \vee (Q \vee R)] \equiv [(P \vee Q) \vee R]$

1. $[P \vee (Q \vee R)] \to [(P \vee Q) \vee R]$	Th. 12, C
2. $[(P \vee Q) \vee R] \to [P \vee (Q \vee R)]$	Th. 12
3. $\{[P \vee (Q \vee R)] \to [(P \vee Q) \vee R]\}\{[(P \vee Q) \vee R] \to [P \vee (Q \vee R)]\}$	DR. 14 (1,2)
4. $[P \vee (Q \vee R)] \equiv [(P \vee Q) \vee R]$	df.

Th. 23) $\vdash P(QR) \equiv (PQ)R$

1. $P(QR) \to (PQ)R$	Th. 10, C
2. $(PQ)R \to P(QR)$	Th. 10
3. $[P(QR) \to (PQ)R][(PQ)R \to P(QR)]$	DR. 14 (1,2)
4. $P(QR) \equiv (PQ)R$	df.

Th. 24) $(P \to Q) \equiv (\sim Q \to \sim P)$

1. $(P \to Q) \to (\sim Q \to \sim P)$	Th. 5
2. $(\sim Q \to \sim P) \vdash (P \to Q)$	DR. 2
3. $(\sim Q \to \sim P) \to (P \to Q)$	D.T.
4. $[(P \to Q) \to (\sim Q \to \sim P)][(\sim Q \to \sim P) \to (P \to Q)]$	DR. 14 (1,3)
5. $(P \to Q) \equiv (\sim Q \to \sim P)$	df.

Th. 25) $[(PQ) \rightarrow R] \equiv [P \rightarrow (Q \rightarrow R)]$

1. $[(PQ) \rightarrow R] \rightarrow [P \rightarrow (Q \rightarrow R)]$	Th. 14
2. $[P \rightarrow (Q \rightarrow R)] \rightarrow [(PQ) \rightarrow R]$	Th. 13
3. $\{[(PQ) \rightarrow R] \rightarrow [P \rightarrow (Q \rightarrow R)]\}\{[P \rightarrow (Q \rightarrow R)] \rightarrow [(PQ) \rightarrow R]\}$	DR. 14 (1,2)
4. $[(PQ) \rightarrow R] \equiv [P \rightarrow (Q \rightarrow R)]$	df.

Th. 26) $\vdash P \equiv PP$

1. $P \rightarrow PP$	Ax. 1
2. $PP \rightarrow P$	Ax. 2
3. $(P \rightarrow PP)(PP \rightarrow P)$	DR. 14 (1,2)
4. $P \equiv PP$	df.

Th. 26, C) $\vdash P \equiv P \vee P$

1. $P \rightarrow (P \vee P)$	Th. 17
2. $P \rightarrow P$	Th. 7
3. $P \rightarrow P$	Th. 7
4. $(P \vee P) \rightarrow P$	DR. 10 (2,3)
5. $[P \rightarrow (P \vee P)][(P \vee P) \rightarrow P]$	DR. 14 (1,4)
6. $P \equiv P \vee P$	df.

Th. 27) $\vdash \sim(PQ) \equiv (\sim P \vee \sim Q)$

1. $(\sim P \vee \sim Q)(\sim P \vee \sim Q) \equiv (\sim P \vee \sim Q)$	Th. 26
2. $(\sim P \vee \sim Q) \equiv (\sim P \vee \sim Q)$	MT IV, C
3. $\sim(\sim\sim P\sim\sim Q) \equiv (\sim P \vee \sim Q)$	df.
4. $\sim\sim P \equiv P$	Th. 19
5. $\sim(P\sim\sim Q) \equiv (\sim P \vee \sim Q)$	MT IV, C
6. $\sim\sim Q \equiv Q$	Th. 19
7. $\sim(PQ) \equiv (\sim P \vee \sim Q)$	MT IV, C

Th. 28) $\vdash \sim(P \vee Q) \equiv (\sim P\sim Q)$

1. $\sim\sim(\sim P\sim Q) \equiv (\sim P\sim Q)$	Th. 19
2. $\sim(P \vee Q) \equiv (\sim P\sim Q)$	df.

Th. 29) $\vdash (P \rightarrow Q) \equiv (\sim P \vee Q)$

1. $(\sim P \vee Q)(\sim P \vee Q) \equiv (\sim P \vee Q)$	Th. 26
2. $(\sim P \vee Q) \equiv (\sim P \vee Q)$	MT IV, C
3. $\sim(\sim\sim P\sim Q) \equiv (\sim P \vee Q)$	df.
4. $\sim\sim P \equiv P$	Th. 19
5. $\sim(P\sim Q) \equiv (\sim P \vee Q)$	MT IV, C
6. $(P \rightarrow Q) \equiv (\sim P \vee Q)$	df.

Th. 30, C) ├ [(P v Q)R] ≡ [PR v QR]

1. PR → RP	Th. 8
2. RP → R	Ax. 2
3. PR → R	DR. 6
4. QR → RQ	Th. 8
5. RQ → R	Ax. 2
6. QR → R	DR. 6
7. (PR v QR) → R	DR. 10 (3,6)
8. PR → P	Ax. 2
9. QR → Q	Ax. 2
10. (PR v QR) → (P v Q)	DR. 9 (8,9)
11. [PR v QR] → [(P v Q)R]	DR. 8 (10,7)
12. [(P v Q)R] → [PR v QR]	Th. 18
13. {[(P v Q)R] → [PR v QR]}{[PR v QR] → [(P v Q)R]}	DR. 14 (12,11)
14. [(P v Q)R] ≡ [PR v QR]	df.

Th. 31') P ≡ Q ├ (PQ) v (~P~Q)

1. P ≡ Q	Premiss
2. ~(PP) ≡ ~P v ~P	Th. 27
3. ~P v ~P ≡ ~P	Th. 26, C
4. ~(PP) ≡ ~P	MT IV, C
5. ~P ≡ ~P~P	Th. 26
6. ~(PP) ≡ ~P~P	MT IV, C (4,5)
7. ~(PQ) ≡ ~P~Q	MT IV, C (6,1)
8. [~(PQ) → ~P~Q][~P~Q → ~(PQ)]	df.
9. ~(PQ) → ~P~Q	DR. 19
10. ~[~(PQ)~(~P~Q)]	df.
11. (PQ) v (~P~Q)	df.

Th. 31") (PQ) v (~P~Q) ├ P ≡ Q

1. (PQ) v (~P~Q)	Premiss
2. PQ → P	Ax. 2
3. ~P~Q → ~Q~P	Th. 8
4. ~Q~P → ~Q	Ax. 2
5. ~P~Q → ~Q	DR. 6 (3,4)
6. (PQ) v (~P~Q) → (P v ~Q)	DR. 9 (2,5)
7. P v ~Q	R 1 (1,6)
8. ~(~P~~Q)	df.
9. ~~Q ≡ Q	Th. 19
10. ~(~PQ)	MT IV, C (8,9)
11. ~PQ ≡ Q~P	Th. 8
12. ~(Q~P)	MT IV, C (10,11)
13. Q → P	df.
14. PQ → QP	Th. 8
15. QP → Q	Ax. 2
16. PQ → Q	DR. 6
17. ~P~Q → ~P	Ax. 2
18. (PQ) v (~P~Q) → (Q v ~P)	DR. 9 (16,17)
19. Q v ~P	R 1 (1,18)
20. ~(~Q~~P)	df.

21. ~~P ≡ P	Th. 19
22. ~(~QP)	MT IV, C (20,21)
23. ~QP ≡ P~Q	Th. 8
24. ~(P~Q)	MT IV, C (22,23)
25. P → Q	df.
26. (P → Q)(Q → P)	DR. 14 (25,13)
27. P ≡ Q	df.

Th. 31) [P ≡ Q] ≡ [(PQ) v (~P~Q)]

1. P ≡ Q ├ (PQ) v (~P~Q)	Th. 31'
2. [P ≡ Q] → [(PQ) v (~P~Q)]	D.T.
3. (PQ) v (~P~Q) ├ P ≡ Q	Th. 31"
4. [(PQ) v (~P~Q)] → [P ≡ Q]	D.T.
5. {[P ≡ Q] → [(PQ) v (~P~Q)]}{[(PQ) v (~P~Q)] → [P ≡ Q]}	DR. 14 (2,4)
6. [P ≡ Q] ≡ [(PQ) v (~P~Q)]	df.

Th. 32') (P v Q)(P v R) ├ P v (QR)

1. (P v Q)(P v R)	Premiss
2. P v Q	DR. 19
3. P v R	DR. 19, C
4. ~(~P~Q)	df. (2)
5. ~(~P~R)	df. (3)
6. ~P → Q	df. (4)
7. ~P → R	df. (5)
8. ~P → (QR)	DR. 8 (6,7)
9. ~[~P~(QR)]	df.
10. P v (QR)	df.

Th. 32) ├ [P v (QR)] ≡ [(P v Q)(P v R)]

1. P → P	Th. 7
2. QR → Q	Ax. 2
3. [P v (QR)] → (P v Q)	DR. 9 (1,2)
4. QR → RQ	Th. 8
5. RQ → R	Ax. 2
6. QR → R	DR. 6
7. [P v (QR)] → (P v R)	DR. 9 (1,6)
8. [P v (QR)] → [(P v Q)(P v R)]	DR. 8 (3,7)
9. (P v Q)(P v R) ├ P v (QR)	Th. 32'
10. (P v Q)(P v R) → [P v (QR)]	D.T.
11. {[P v (QR)] → [(P v Q)(P v R)]}{[(P v Q)(P v R)] → [P v (QR)]}	DR. 14 (8,10)
12. [P v (QR)] ≡ [(P v Q)(P v R)]	df.

8.6 Deductive Completeness

Metatheorem VI (pg. 252)

Let P_1, P_2, P_3,..., P_n be any wffs and let Q and R be any two wffs constructed out of them by means of 'v'. If each P_i ($1 \leq i \leq n$) occurs exactly once in each of the wffs Q and R, then $\vdash Q \equiv R$.

Proof: We will use strong induction on the number of 'factors' (disjuncts) P_i in Q and R.

α) If n = 1, then Q and R are identical, each being the same wff, P_1. So, $\vdash Q \equiv P_1$ and $\vdash R \equiv P_1$.
Since $\vdash P_1 \equiv P_1$ by Th. 7 and DR. 14, we have that $\vdash Q \equiv R$ by MT IV, C. applied twice.

β) Now we assume Metatheorem VI holds true for every k < n factors P_1, P_2, P3,..., P_k.
Now consider Q and R, each constructed out of n > 1 factors P_1, P_2, P_3,..., P_n.
We now let Q be S v T and we let R be X v Y.
Each of the wffs S and T must contain at least one of the factors P_1, P_2, P_3,..., P_n.
We can assume that P_1 is a factor of S, because if it were not a factor, then by Th. 20 and a relabeling of S and T, we would obtain $\vdash Q \equiv S \text{ v } T$, where P_1 is a factor of S.
Since T contains at least one of P_2, P_3,..., P_n as a factor, S contains < n of the factors P_1, P_2, P_3,..., P_n.
Thus, either S is P_1 and $\vdash Q \equiv P_1 \text{ v } T$, or by the construction of Q, we have that $\vdash S \equiv P_1 \text{ v } S'$, where S' is a wff that contains all the factors of S except P_1.
In this latter case, by MT IV, C. we have that $\vdash Q \equiv (P_1 \text{ v } S') \text{ v } T$.
Then by Th. 22 and MT IV, C. we obtain $\vdash Q \equiv P_1 \text{ v } (S' \text{ v } T)$.
In either case, we can say that there exists some wff, call it T', such that $\vdash Q \equiv P_1 \text{ v } T'$.
(In the former case T' would be T, while in the latter case T' would be S' v T)
By the same chain of reasoning we can show that there exists some wff, call it Y', such that $\vdash R \equiv P_1 \text{ v } Y'$.
Since both T' and Y' contain n - 1 < n factors P_2, P_3,...,P_n, by the induction hypothesis, $\vdash T' \equiv Y'$.
As we saw in the (α) case, $\vdash P_1 \equiv P_1$. Thus, by DR. 16, C. we have that $\vdash P_1 \text{ v } T' \equiv P_1 \text{ v } Y'$.
Finally, by MT IV, C. applied twice, we obtain $\vdash Q \equiv R$.
Ergo, the induction holds and Metatheorem VI follows by strong induction.