Dispersion measurements with minimum and maximum deviated beams

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(Received 21 December 2005; accepted 5 May 2006)

The prism spectrometer has been the standard apparatus for index of refraction measurements and a typical instrument in undergraduate laboratories for dispersion experiments. Because much care must be employed when aligning a prism spectrometer, a more robust method for measuring index of refraction is needed. By using a laser, prism, rotating platform, and tape measure, we can measure the index of refraction to five significant figures. Due to its ease of use, undergraduates can make more accurate measurements with this method than by using a poorly aligned prism spectrometer. Although a tunable laser is preferable for easily changing wavelengths, two laser pointers, one red and one green, may be used, allowing for an inexpensive method of accurately obtaining the constants in the two-term Cauchy equation relating the index of refraction to the wavelength. © 2006 American Association of Physics Teachers.

DOI: 10.1119/1.2209245

I. INTRODUCTION

Determining the wavelength dependence of the index of refraction is a common and important measurement in undergraduate optics laboratories. The prism spectrometer is most frequently used for making these measurements.\textsuperscript{1,2} It is also the most accurate method for determining the index of refraction, with an uncertainty of \( \pm (5 \times 10^{-5}) \) (Refs. 2 and 3). The initial alignment of the prism spectrometer is very time consuming for students. Although the time spent on alignment is a valuable laboratory experience, students' time is better spent exploring the physical results of the experiment. Maintaining and verifying the alignment of the spectrometer for large numbers of users requires frequently repeating the alignment procedure or collecting data with a poorly aligned instrument.

Methods other than the prism spectrometer have been developed for measuring dispersion in undergraduate laboratories.\textsuperscript{4-8} We report a simple and conceptually straightforward approach using the angle of minimum deviation and the angle of maximum deviation for measuring the wavelength dependence of the index of refraction of transparent materials. The advantages of this method are easy implementation, conceptual simplicity, the possibility of an inexpensive apparatus, and high accuracy of the results.

II. THEORY

When a beam of light passes through a prism, it undergoes refraction upon entering and exiting the prism, as indicated in Fig. 1. When monochromatic light undergoes the minimum deviation \( \delta_{\text{min}} \), a simple equation relates \( \delta_{\text{min}} \), the apex angle of the prism \( \alpha \), and the prism’s index of refraction \( n \) (Ref. 9),

\[
n(\lambda) = \frac{\sin[(\delta_{\text{min}}(\lambda) + \alpha)/2]}{\sin(a/2)}.
\]

The angular deflection of the beam is measured, and for a known apex angle \( \alpha \), the index of refraction \( n(\lambda) \) at this wavelength can be obtained. Due to the high intensity and low divergence of a laser beam, the deviation of other beams may also be measured and used to determine \( n(\lambda) \). We may easily observe the six beams that emanate from an equilateral prism as shown in Fig. 1: beam 1 is reflected at the first interface at point \( a \); beam 2, after passing through the first interface, is refracted into the air at point \( b \); the angle of this beam’s deflection is used in Eq. (1); beam 3 is internally reflected at point \( b \), and the angle \( \delta_{\text{R3}} \) is measured; beam 4 is internally reflected at point \( c \), and the angle \( \delta_{\text{R4}} \) is measured. Beam 5 is internally reflected at point \( a' \), and beam 6 is reflected at point \( b' \) and refracted at the air at point \( c' \). The light reflected at point \( c' \) will eventually follow all the other beams after interacting with the front surface at point \( a \).

Beam 4 in Fig. 1 undergoes a maximum deflection at the same prism orientation that beam 2 experiences a minimum deflection. Beam 3 also undergoes an extreme deflection at this prism orientation. We can relate the deflection of any of these beams to the index of refraction, thus making three independent measurements of the index of refraction.

A significant simplification to the ray diagram and an aid to making the measurement more transparent occur when the incident light is centered on an equilateral prism. For this case, when the prism is oriented such that beam 2 undergoes a minimum angle of deflection, points \( a \) and \( a' \), \( b \) and \( b' \), and \( c \) and \( c' \) converge, while the corresponding refracted and reflected beams overlap (see Fig. 2). In addition, the relation between the index of refraction and the maximum angular deflection \( \varphi_{\text{max}} \) of beam 4 simplifies to

\[
n(\lambda) = \frac{\cos[\varphi_{\text{max}}(\lambda)/2]}{\sin(30^\circ)}.
\]

III. EXPERIMENTAL PROCEDURE

Light from a tunable krypton-argon ion laser (Omnicrome Series 43) is incident on an equilateral prism that is mounted on a rotation stage (see Fig. 3). The following wavelengths were used: 676.4, 647.1, 568.2, 530.9, 520.8, 514.5, 496.5, 488.0, and 476.2 nm. The laser beam is oriented so that it is reasonably perpendicular to a laboratory wall. One method for this alignment is to hold a sufficiently large mirror (10 cm \( \times \) 10 cm) against the wall and orient the incident laser beam so that the back reflected beam overlaps it. Using a larger mirror, in contrast to one of the standard, small (diam = 2.5 cm) laser mirrors, is important to compensate for small, local variations in the orientation of the wall.
Once the beam is perpendicular to the wall, the location of the undeflected beam should be marked on the wall by taping a piece of paper onto the wall and placing a pencil mark where the beam makes a spot on the paper. Then the prism, which is mounted on a rotation stage, is inserted into the beam. For this technique it is important to place the center of the prism’s front face directly above the axis of the rotation stage and ensure that the laser beam is incident on the vertical symmetry plane of the prism face. Centering is made easy by placing a mask with a small hole in the center over the prism face and aligning the prism so that the beam passes through the mask.

The prism is rotated until a minimum deflection of beam 2 is achieved. As another check, when the angle of minimum deflection occurs, beams 2 and 5 should completely overlap. At the same prism orientation a maximum deflection occurs for beam 4, and beam 1 completely overlaps beam 4. Appropriate marks on the wall should identify the locations of the spots for the minimally deflected beam 2 and the maximally deflected beam 4. The prism should next be rotated switching beams 2 and 4 symmetrically about the incident beam. With a measuring tape mounted to the wall, these measurements can be done quickly.

In Fig. 3 there are two symmetrical orientations of the prism for which the minimum deflection of the beam is achieved. For ideal alignment the distance between the two minimally deviated beams is twice the deflection \( d \) of each minimally deflected beam. If the laser beam is perfectly perpendicular to the wall, the angle of minimum deflection is related to the distance of the refracted beam spot from its undeflected location and the distance \( \ell \) of the intersection of the undeflected and refracted beams from the wall,

\[
\tan \delta_{\text{min}} = \frac{d}{\ell}.
\]

If the laser beam is not perfectly perpendicular to the wall, but deviates by \( \epsilon \), \( \delta_{\text{min}} \) can be found from the law of sines,

\[
\sin \delta_{\text{min}} \cos(\delta_{\text{min}} + \epsilon) = \frac{d}{\ell}.
\]

The index of refraction for a particular wavelength of light can be determined from Eq. (1) if the apex angle \( \alpha \) of the prism is known. The apex angle, which for these methods is constrained to be 60°, may be given to the students or may be measured by them.

Any of the refracted beams can be used to measure the index of refraction. In this study the distance between where the maximally deviated beam 4 intersects the wall for the two prism orientations was also used for the analysis. A simple geometrical consideration leads to a relation between the maximum deflection angle \( \varphi_{\text{max}} \) and the minimum deflection angle,

\[
\varphi_{\text{max}} = 120^\circ - \delta_{\text{min}}.
\]

Equation (5) together with Eq. (1) may be used to derive Eq. (2), the relation between the index of refraction and the maximum deflection angle. Similarly, this angle is related to the measured distance \( d' \) of the deflected beam spot from its undeflected location and the distance \( \ell' \) of the intersection of the undeflected and deflected beams from the wall by

\[
\frac{\sin \varphi_{\text{max}}}{\cos(\varphi_{\text{max}} - \epsilon)} = \frac{d'}{\ell'}.
\]

IV. ERROR ANALYSIS

The error associated with the uncertainty in the prism apex angle may be neglected because for commercially distributed prisms it is about \( \pm 2 \) arc min. The error \( \Delta n \) in the determi-
nation of the index of refraction essentially results from the
uncertainty in the measurement of the deflection angles. The
error analysis for both the minimally and maximally de-
flected beams is similar and our discussion is limited to the
former. For the beam that undergoes minimum deviation,
\[ \Delta n(\lambda) = \frac{\cos \left( \delta_{\min} + 60^\circ \right) \Delta \delta_{\min}}{2 \sin(30^\circ)} \]  
(7)
The error in the determination of the deflection angle results
from the accuracy of measurements of the distances \( d \) and \( \ell \),
as well as the accuracy of aligning the laser beam perpen-
dicularly to the wall. For a single deflection angle (Fig. 3) the
error in the determination of the deflection angle is
\[ \Delta \delta_{\min} = \sqrt{\left( \frac{d^2}{\ell^2} \epsilon^2 \right)^2 + \left( \frac{\ell}{\ell^2 + d^2} \right)^2 (\Delta d)^2 + \left( \frac{d}{\ell^2 + d^2} \right)^2 (\Delta \ell)^2}, \]  
(8)
which depends on the deviation from the normal to the wall.
Note that by increasing the distance from the prism to the
wall, the effect of the last two terms decreases. However, the
error resulting from the deviation from perpendicularity is
not affected by this procedure.

Although we could arrive at \( d \) by measuring the distance
between one of the minimally deflected beams and the unde-
flected beam, the measurement of the distance between the
two deflected beams will eliminate the error due to non-
normal incidence \( (\epsilon \neq 0) \). The uncertainty of the deflection
angle obtained from the measurement of the deflections pro-
duced in both prism orientations reduces to
\[ \Delta \delta_{\min} = \frac{1}{\ell^2 + d^2} \sqrt{\left( \frac{\ell}{\ell^2 + d^2} \Delta \ell \right)^2 + \left( \frac{d}{\ell^2 + d^2} \Delta d \right)^2}, \]  
(9)
where the bar above a symbol refers to the quantities ac-
quired from experiments in which the average deflection is
determined. The decrease in the uncertainty that results from
using the average deflection is confirmed in Fig. 4, which
shows the index of refraction obtained by single deflection
measurements and double deflection measurements.

In this approach the error in the determination of the de-
flection angle calculated from Eq. (7) can be reduced by
placing the prism further from the wall (increasing \( \ell \)). Prac-
tical considerations, such as the size of the room, place a
limit on the maximum prism-wall separation. An approxi-
mate 70 cm distance between the prism and the wall results
in a beam displacement (for \( n=1.8 \)) of around 380 cm,
which yields uncertainties of about 0.0003 in the index of
refraction. This uncertainty is in good agreement with the
difference between the measured and accepted values (see
Table I).

V. DISCUSSION

Representative experimentally determined values of the
index of refraction \( n(\lambda) \) using both the minimally and maxi-
mally deviated beams for an equilateral prism made from
SF11 glass (Edmund Scientific, part # G47-284) are shown
in Table I and Fig. 4, and compared to accepted values. The
accepted values were determined by fitting the three-term
Cauchy formula to the manufacturer supplied values of the
index of refraction (see Table II) and then using the Cauchy
formula to obtain \( n(\lambda) \) at the desired wavelengths:9
\[ n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}. \]  
(10)
The fit to the manufacturer’s data yields \( A=1.7447(1), \)
\( B=1.21(1) \times 10^{-14} \text{ m}^2, \) and \( C=5.9(1) \times 10^{-28} \text{ m}^4. \) The
Cauchy formula was also fit to the experimental data. The
results are shown in Table II and offer another way of dem-
onstrating the accuracy of this method.

Most of the expense for this experiment is for the tunable
laser. The cost can be greatly reduced by using a red and a
green laser diode in place of the tunable laser. Apertures can
be used to insure the laser beams are properly aligned and an
inexpensive wavelength meter can be built to accurately de-
termine the wavelength of each laser.10 With this data, the

<table>
<thead>
<tr>
<th>( \lambda ) (nm)</th>
<th>( n ) (minimum deviation)</th>
<th>( n ) (maximum deviation)</th>
<th>( n ) (accepted)</th>
</tr>
</thead>
<tbody>
<tr>
<td>676.4</td>
<td>1.77415</td>
<td>1.77414</td>
<td>1.77397</td>
</tr>
<tr>
<td>647.1</td>
<td>1.77720</td>
<td>1.77714</td>
<td>1.77696</td>
</tr>
<tr>
<td>568.2</td>
<td>1.78806</td>
<td>1.78795</td>
<td>1.78784</td>
</tr>
<tr>
<td>530.9</td>
<td>1.79513</td>
<td>1.79529</td>
<td>1.79506</td>
</tr>
<tr>
<td>520.8</td>
<td>1.79737</td>
<td>1.79750</td>
<td>1.79733</td>
</tr>
<tr>
<td>514.5</td>
<td>1.79910</td>
<td>1.79901</td>
<td>1.79883</td>
</tr>
<tr>
<td>496.5</td>
<td>1.80362</td>
<td>1.80364</td>
<td>1.80349</td>
</tr>
<tr>
<td>488.0</td>
<td>1.80601</td>
<td>1.80618</td>
<td>1.80591</td>
</tr>
<tr>
<td>476.2</td>
<td>1.80964</td>
<td>1.80990</td>
<td>1.80953</td>
</tr>
</tbody>
</table>

Table II. Values of the Cauchy constants \( A, B, \) and \( C. \) The first column is a
fit of \( n(\lambda)=A+\frac{B}{\lambda^2}+\frac{C}{\lambda^4} \) to the manufacturer’s data from 436 to 707 nm. The
second column is the same fit but over 480–656 nm; the third column is a fit
to the data collected in this experiment.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Accepted (436–707 nm)</th>
<th>Accepted (480–656 nm)</th>
<th>Calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>1.7454(3)</td>
<td>1.7447(1)</td>
<td>1.7450 (6)</td>
</tr>
<tr>
<td>( B )</td>
<td>1.16(2) \times 10^{-14}</td>
<td>1.21(1) \times 10^{-14}</td>
<td>1.20(2) \times 10^{-14}</td>
</tr>
<tr>
<td>( C )</td>
<td>6.7(3) \times 10^{-28}</td>
<td>5.9(1) \times 10^{-28}</td>
<td>5.9(2) \times 10^{-28}</td>
</tr>
</tbody>
</table>
constants $A$ and $B$ in Eq. (10) with $C=0$ can be determined and the spectrum of the index of refraction obtained over a range of wavelengths.

**VI. SUGGESTED PROBLEMS**

Once the wavelength dependence of the index of refraction of the prism is known (either measured as described here or from the manufacturer’s specifications), the system may be used to measure the wavelength of a laser beam. Students could measure the wavelength of a number of laser pointers to determine the variation in the wavelength of nominally identical lasers.

If an infrared diode laser and either an infrared camera or infrared detector card are available for the experiment, students could extend their measurements of the index of refraction beyond the visible spectrum.

Although beam 3 (Fig. 1) was noted in the text, no equation was given relating its angle of deviation to the index of refraction. Students can derive this equation, along with the uncertainty in the index of refraction. They may include measurements from this beam along with those from the minimally and maximally deviated beams.

**VII. CONCLUSION**

By taking advantage of inexpensive lasers, sending the light through an equilateral prism, and using reasonable distances to accurately determine deviation angles, the wavelength dependence of the index of refraction of light can be measured to five significant figures. Students find this method of measuring dispersion to be much easier to understand than the prism spectrometer because there are neither lens systems nor complicated alignment procedures.

**ACKNOWLEDGMENTS**

We gratefully acknowledge Mr. Les Porter for technical assistance and an anonymous reviewer who made very constructive suggestions.


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**THE INTUITION OF SPACE AND TIME**

The core idea of Einstein’s rebuilding of our intuition is that we must stop thinking about “space and time,” which is something that is “given to us,” and must instead think about “measuring positions and times,” which is something we do. Only our measurements have real existence. We build up an intuition of something called space and time, which we believe exists beyond these measurements and which would be there even if the measurements were never taken. Einstein’s first commandment was to pay attention only to your measurements and worry later about the properties of the more abstract notion of space and time.