Homework 1

problems: 1.2, 1.21, 1.23, 1.53
Problem 1.2

The standard kilogram is a platinum-iridium cylinder 39.0 mm in height and 39.0 mm in diameter. What is the density of the material?

According to the definition of density \( \rho \), the (differential) mass \( dm \) of a differential fragment of the cube is proportional to the volume \( dV \)

\[
dm = \rho dV.
\]

For a uniform object the integration of the above function is very simple. As the density is a “constant function” it can be factored out of the integral. By the definition, the remaining integral represents the volume \( V \) of the cylinder, which is related to the size of the cylinder (height \( h \) and diameter \( 2r \)) by a well-known relation

\[
V = \pi r^2 \cdot h.
\]

Hence the mass of the cylinder and its dimensions are related by

\[
m = \int dm = \int \rho dV = \rho \int dV = \rho V = \rho \cdot \pi r^2 \cdot h
\]

Solving the above equation for the unknown density leads to the answer

\[
\rho = \frac{m}{\pi r^2 \cdot h} = \frac{1 \text{ kg}}{\pi \left(19.5 \times 10^{-3} \text{ m}\right)^2 \left(39 \times 10^{-3} \text{ m}\right)} \approx \frac{2.15 \text{ kg}}{\text{m}^3}
\]

Note. When the height and the diameter of a cylinder are equal, the cylinder has the smallest possible surface area and the given volume.
Problem 1.21

One cubic meter of aluminum has a mass of $2.70 \times 10^3$ kg, and one cubic meter of iron has a mass of $7.86 \times 10^3$ kg. Find the radius of a solid aluminum sphere that balances a solid iron sphere of radius 2.00 cm on the equal-arm balance.

\begin{align*}
\text{a)} & \quad \text{The text provides information about the density } \rho \text{ of both substances. From the given mass of one cubic foot of each substance we find their values to be} \\
\rho_{\text{Al}} &= \frac{2.7 \times 10^3 \text{kg}}{1 \text{m}^3} = 2.7 \times 10^3 \text{ kg/m}^3 \\
\rho_{\text{Fe}} &= \frac{7.86 \times 10^3 \text{kg}}{1 \text{m}^3} = 7.86 \times 10^3 \text{ kg/m}^3
\end{align*}

With a well-known expression for the volume of a sphere

\[ V = \frac{4}{3} \pi r^3, \]

the equal mass of the two (uniform) spheres requires that

\[ \frac{4}{3} \pi r_{\text{Al}}^3 \cdot \rho_{\text{Al}} = \frac{4}{3} \pi r_{\text{Fe}}^3 \cdot \rho_{\text{Fe}} \]

Hence, the radius of the aluminum sphere should be

\[ r_{\text{Al}} = r_{\text{Fe}} \cdot \frac{\sqrt[3]{\rho_{\text{Fe}}}}{\sqrt[3]{\rho_{\text{Al}}}} = 2.0 \text{ cm} \cdot \frac{\sqrt[3]{7.86 \times 10^3 \text{kg}}}{\sqrt[3]{2.7 \times 10^3 \text{kg}}} \approx 2.86 \text{ cm} \]
Problem 1.23

One gallon of paint (volume = $3.78 \times 10^{-3}$ m$^3$) covers an area of 25.0 m$^2$. What is the thickness of the paint on the wall?

a) For the sake of simplicity we can assume that the paint was uniformly spread over a single wall. The paint forms a column with the height equal to the thickness $t$ of the paint and the base area equal to the area $A$ of the painted surface. Obviously, the volume of this column must be equal to the volume $V$ of the paint

\[ V = At \]

In this equation the thickness is not known. Solving for it, we find the answer

\[ t = \frac{V}{A} = \frac{3.78 \times 10^{-3} \text{ m}^3}{25 \text{ m}^2} = 1.51 \times 10^{-4} \text{ m} \approx 151 \mu\text{m} \]
Problem 1.53

A high fountain of water is located at the center of a circular pool as shown in the figure. Not wishing to get his feet wet, a student walks around the pool and measures its circumference to be 15.0 m. Next, the student stands at the edge of the pool and uses a protractor to gauge the angle of elevation of the top of the fountain to be $55^\circ$. How high is the fountain?

The circumference $l$ of the circular pool is proportional to its radius $r$

$$ l = 2\pi r $$

From the definition of the tangent function, the height of the fountain ($h$), the radius of the fountain and the angle of elevation are related

$$ \tan \alpha = \frac{h}{r} $$

Solving simultaneously (by substitution) the above two equations for the height of the fountain, we find the answer

$$ h = r \cdot \tan \alpha = \frac{l}{2\pi} \cdot \tan \alpha = \frac{15 \text{ m}}{2\pi} \cdot \tan 55^\circ \approx 3.41 \text{ m} $$