Homework 3

problems: 4.5, 4.31, 4.49, 4.67
Problem 4.5

The vector position of a particle varies in time according to the expression
\[ \mathbf{r} = (3.00 \cdot \hat{i} - 6.00t^2 \cdot \hat{j}) \text{ m}. \] (a) Find expressions for the velocity and acceleration as functions of time. (b) Determine the particle’s position and velocity at \( t = 1.00 \text{ s} \).

Note. The position function is not defined unambiguously. It is not clear what the time units are. By default, one should assume that seconds but inserted without the unit into the function. It would be clearer, if the function were presented in the following way
\[ \mathbf{r}(t) = (3.00 \text{ m}) \cdot \hat{i} - \left( 6.00 \frac{\text{m}}{s^2} \cdot t^2 \right) \cdot \hat{j} \]

a) By definition, the velocity function is equal to the derivative of the position function. Assuming a fixed reference frame (with time independent unit vectors)
\[ \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt} \left( 3.00m \cdot \hat{i} - 6.00 \frac{m}{s^2} \cdot t^2 \cdot \hat{j} \right) = 0 \cdot \hat{i} - 12 \frac{m}{s^2} \cdot t \cdot \hat{j} = -12 \frac{m}{s^2} t \cdot \hat{j} \]

By definition, the acceleration function is equal to the derivative of the velocity function.
\[ \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left( -12 \frac{m}{s} \cdot t \cdot \hat{j} \right) = -12 \cdot \hat{j} \frac{m}{s^2} \]

b) The above functions allow one to determine specific values at an arbitrary instant. Hence at \( t = 1 \text{ s} \), the particle is at
\[ \mathbf{r}(1s) = (3.00 \text{ m}) \cdot \hat{i} - \left( 6.00 \frac{m}{s^2} \cdot (1s)^2 \right) \cdot \hat{j} = (3.00 \cdot \hat{i} - 6.00 \cdot \hat{j}) \text{ m} \]

The velocity of the particle at this instant is
\[ \mathbf{v}(1s) = -12 \frac{m}{s^2} \cdot 1s \cdot \hat{j} = -12 \frac{m}{s} \cdot \hat{j} \]
Problem 4.31

A train slows down as it rounds a sharp horizontal curve, slowing from 90.0 km/h to 50 km/h in the 15.0 s that is takes to rounds the curve. The radius of the curve is 150 m. Compute the acceleration at the moment the strain speed reaches 50 km/h. Assume the train slows down at a constant rate during the 15-s interval

In a circular motion Cartesian scalar component of acceleration are trigonometric functions of time. From the information given, it is more convenient to determine the tangential (along the direction of the trajectory) and centripetal (transverse to the trajectory) components of the acceleration.

In any type of motion, the tangential component of acceleration is equal to the rate at the speed of the particle changes. Since in this problem it is assumed that the speed changes at a constant rate, tangential acceleration at an instant under consideration is equal to its average value (derivative of the function is equal to the difference quotient). Hence

\[ a_t(t) = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{(50 - 90) \text{ km/h}}{15 \text{ s}} = \frac{(50 - 90) \text{ km/h} \cdot \frac{10^3 \text{ m/km}}{3600 \text{ s/h}}}{15 \text{ s}} = -0.74 \text{ m/s}^2 \]

The transverse component of acceleration affects only the direction of velocity. In a circular motion, the transverse component is always directed toward the center of the circular path of the particle and is referred as the centripetal acceleration. In a circular motion, the centripetal acceleration depends only on the speed of the object and the radius of path’s curvature.

\[ a_c(t) = \frac{v^2}{r} = \left( \frac{50 \text{ km/h}}{150 \text{ m}} \right)^2 = \left( \frac{50 \text{ km/h} \cdot \frac{10^3 \text{ m/km}}{3600 \text{ s/h}}}{150 \text{ m}} \right)^2 = 1.29 \text{ m/s}^2 \]

The word “acceleration” often refers to the magnitude of the acceleration vector, therefore one may include this meaning also into the answer

\[ a_f = \sqrt{a_t^2 + a_c^2} = \sqrt{(-0.74 \text{ m/s}^2)^2 + (-1.29 \text{ m/s}^2)^2 + (0 \text{ m/s}^2)^2} = 1.49 \text{ m/s}^2 \]
Problem 4.39

Heather Lisa in her Lamborghini accelerates at rate of \( (3\hat{i} - 2\hat{j}) \) m/s\(^2\), while Jill in her Jaguar accelerates at \( (\hat{i} + 3\hat{j}) \) m/s\(^2\). They both start from the rest at the origin on a xy coordinate system. After 5 s, (a) what is Lisa’s speed with respect to Jill, (b) how far apart are they, and (c) what is Lisa’s acceleration relative to Jill?

a) The initial position and initial velocity of both cars are given in the reference frame of the origin. Additionally, the acceleration of each car is given in this reference frame. Using inverse relations the velocity and the position functions can be found for both cars. In the reference frame of the origin, Jill’s velocity at time \( t \) is

\[
u(t) = v_0 + \int_0^t a_J dt' = [1,3] \frac{m}{s^2} \cdot t
\]

while Heather’s velocity in this reference frame is

\[
v(t) = v_0 + \int_0^t a_H dt' = [3,-2] \frac{m}{s^2} \cdot t
\]

In the reference frame of the origin Jill’s position at time \( t \) is

\[
R(t) = R_0 + \int_0^t u(t) dt' = \frac{1}{2} [1,3] \frac{m}{s^2} \cdot t^2
\]

while Heather’s position in this reference frame is

\[
r(t) = r_0 + \int_0^t v(t') dt' = \frac{1}{2} [3,-2] \frac{m}{s^2} \cdot t^2
\]

Looking at the figure, Heather’s position with respect to Jill is related to the positions of both girls with respect to the origin. With the appropriate choice of Jill’s coordinate system

\[
r'(t) = r(t) - R(t) = \frac{1}{2} [3,-2] \frac{m}{s^2} \cdot t^2 - \frac{1}{2} [1,3] \frac{m}{s^2} \cdot t^2 = \frac{1}{2} [2,-5] \frac{m}{s^2} \cdot t^2
\]
a) Directly from the definition of velocity, Heather’s velocity with respect to Jill is described by the following function

\[ \mathbf{v}'(t) = \frac{d\mathbf{r}'}{dt} = \left[ 2, -5 \right] \frac{m}{s^2} \cdot t \]

At \( t_1 = 5 \) s, the velocity is therefore

\[ \mathbf{v}'(5s) = \left[ 2, -5 \right] \frac{m}{s^2} \cdot 5s = \left[ 10, -25 \right] \frac{m}{s} \]

and from the definition of speed its value is

\[ v' = \sqrt{10^2 + (-25)^2} \frac{m}{s} = 26.9 \frac{m}{s} \]

b) At the considered instant, Heather’s position with respect to Jill is

\[ \mathbf{r}'(5s) = \frac{1}{2} \cdot \left[ 2, -5 \right] \frac{m}{s^2} \cdot (5s)^2 = \left[ 25, -62.5 \right] m \]

Magnitude of the position vector represents the distance of the object from the origin of the reference frame. Therefore the distance from Jill to Heater is

\[ r' = \sqrt{25^2 + (-62.5)^2} m = 67.3 m \]

c) Directly from the definition, Heather’s acceleration in Jill’s reference frame is

\[ \mathbf{a}'(t) = \frac{d\mathbf{v}'}{dt} = \left[ 2, -5 \right] \frac{m}{s^2} \]
Problem 4.59

A skier leaves the ramp of a ski jump with a velocity of 10 m/s, 15° above the horizontal, as in figure 4.28. The slope is inclined at 50°, and air resistance is negligible. Find (a) the distance from the ramp that to where the jumper lands and (b) the velocity components just before landing. (How do you think the result might be affected if air resistance were included? Note that jumpers lean forward in the shape of an airfoil with their hands at their sides to increase their distance. Why does it work?)

In order to solve the problem I will first describe the motion of the skier. This function will allow me to determine the instant when the skier jumps and lands, and determine the initial and final position, from which the magnitude of the displacement can be found. Finally I can find the velocity at the instant of landing.

Let's specify the reference time and the reference frame in which I will describe the motion. I decided to choose the instant when the skier, treated as a particle, leaves the ramp as the reference instant and the end of the ramp as the origin of a Cartesian system. The axes are in the horizontal and vertical direction. The z-axis is perpendicular to the motion, therefore we know that the components of all vectors related to the motion are zero in the z-direction. We can therefore describe the motion by two dimensional vectors $\mathbf{R}^2$.

With the above assumptions the initial position and initial velocity are:

$$ \mathbf{r}_0 = [0, 0] \text{m}, \quad \mathbf{v}_0 = v_0 [\cos \alpha, \sin \alpha] = \begin{bmatrix} 9.7 \text{ m/s} \\ 2.6 \text{ m/s} \end{bmatrix}. $$

With good approximation we can assume that the skier moves with constant (free fall) acceleration
This information allows us to predict the skier's velocity and position at any instant \( t \).

\[
\overline{a}(t) = \overline{g} = \begin{bmatrix} 0, -9.8 \frac{m}{s^2} \end{bmatrix}
\]

\[\overline{v}(t) = \overline{v}_0 + \int_{t_0}^{t} \overline{a}(t')dt' = \overline{v}_0 + \frac{\overline{g}t}{2} = \begin{bmatrix} 9.7 \frac{m}{s} \cdot t, 2.6 \frac{m}{s} - 4.9 \frac{m}{s^2} \cdot t \end{bmatrix} \]

\[\overline{r}(t) = \overline{r}_0 + \int_{t_0}^{t} \overline{v}(t')dt' = \overline{r}_0 + \overline{v}_0 t + \frac{1}{2} \overline{g}t^2 = [0, 0]m + [9.7, 2.6] \frac{m}{s} \cdot t + [0, -4.9] \frac{m}{s^2} \cdot t^2 =\]

\[\begin{bmatrix} 9.7 \frac{m}{s} \cdot t, 2.6 \frac{m}{s} \cdot t - 4.9 \frac{m}{s^2} \cdot t^2 \end{bmatrix}\]

When the skier lands, his or her position is somewhere along the slope of the mountain. Therefore, in this coordinate system the components of his or her landing position must satisfy

\[\tan \beta = \frac{-y_1}{x_1}\]

Use equation (2) to find out if this is possible. There must be such an instant \( t_1 \) that

\[\tan \beta = \frac{-v_{y0} t_1 + \frac{1}{2} g \cdot t_1^2}{v_{x0} \cdot t_1}\]

This equation has the solution

\[t_1 = 2 \cdot \frac{v_{x0} \tan \beta + (v_{y0})}{g} = 2 \cdot \frac{9.7 \frac{m}{s} \cdot \tan 50^\circ \beta + 2.5 \frac{m}{s}}{9.8 \frac{m}{s^2}} \approx 2.87s\]

a) At this instant the position of the skier (the landing location) is

\[\overline{r}(2.87s) = \begin{bmatrix} 9.7 \frac{m}{s} \cdot 2.87s, 2.6 \frac{m}{s} \cdot 2.87s - 4.9 \frac{m}{s^2} \cdot (2.87s)^2 \end{bmatrix} = [27.9m, -32.9m]\]

The displacement is defined in such a way that its magnitude is equal to the distance \( \Delta r \) between the initial and the final positions. Therefore

\[\Delta r = |\overline{r}_f - \overline{r}_0| = |x_1, y_1| - [0, 0] = \sqrt{x_1^2, y_1^2} = \sqrt{(27.9m)^2 + (-32.9m)^2} = 43.1m\]
b) Equation (1) allows one to find the velocity of the skier at any instant that he/she is in the air. Just before the landing, his or her velocity is therefore

\[ \mathbf{v}(t) = \left[ 9.7 \frac{\text{m}}{\text{s}}, 2.6 \frac{\text{m}}{\text{s}} - 9.8 \frac{\text{m}}{\text{s}^2} \cdot 2.87 \text{s} \right] = \left[ 9.7 \frac{\text{m}}{\text{s}}, -28.1 \frac{\text{m}}{\text{s}} \right] \]

The presence of interaction with the air makes the competition more exciting. The skiers can influence the length of the jump. It would be surprising but theoretically jumps can be longer in the presence of interaction with the air! The idea is to reduce the y-component of the acceleration by using air as a source of an additional force (lift), while maintaining almost constant horizontal component of the velocity. The principle is applicable in hand gliding (when there is no air convection). Some animals also learned how to increase the range of their jumps by taking advantage of interaction with the air.