Homework 11

Problems: 20.37, 22.33, 22.41, 22.67
Problem 20.37

A 1.00-kg block of aluminum is heated at atmospheric pressure such that its temperature increases from 22.0°C to 40.0°C. Find (a) the work done by the aluminum, (b) the energy added to it by heat, and (c) the change in its internal energy.

\[ m = 1 \text{ kg} \]

\[ T_i = 22 \degree \text{C}; \quad T_f = 40 \degree \text{C} \]

\textbf{a)} The macroscopic work done by a system is defined by the change in the system's volume

\[ dW = PdV. \]

The stress (hydrostatic pressure) in the block is equal to the external pressure of 1 atm. Since, in this process, the pressure is constant, it is easy to find the work (integral)

\[ \Delta W = \int dW = P \Delta V \]

The thermal expansion of the block causes a change in volume. From the definition of the volumetric coefficient of expansion we can relate the change in volume to the change in temperature. Assuming that the change in volume is small when compared with the initial volume we can use the approximate relation

\[ \Delta V = \beta V \Delta T = 3\alpha V \Delta T \]

We can express the volume in terms of the mass of the block, using the definition of density. For a uniform object integration of the density leads to a simple relation
\[ m = \rho V \]

*Work done in the process is therefore*

\[ \Delta W = P \cdot 3\alpha \cdot \frac{m}{\rho_{Al}} \cdot \Delta T = \]

\[ = 1.013 \text{Pa} \cdot 3 \cdot 24 \cdot 10^{-6} \text{K}^{-1} \cdot \frac{1 \text{kg}}{2.7 \cdot 10^3 \text{kg/m}^3} \cdot (40^\circ - 22^\circ) = 48 \text{mJ} \]

*b) The process is isobaric and the specific heat in this temperature interval does not depend on temperature. The integration of heat is therefore easy to perform*

\[ \Delta Q = \int_{T_i}^{T_f} \text{dQ} = \int_{\text{process}} \text{mc}dT = mc\Delta T = \]

\[ = 1 \text{kg} \cdot 900 \frac{\text{J}}{\text{kg} \cdot \text{K}} \cdot (40 - 22)\text{K} = 16.2 \text{kJ} \]

c) *From the first law of thermodynamics, the change in the internal energy of the block is*

\[ \Delta U = \Delta Q - \Delta W = 16.2 \text{kJ} - 0.48 \text{mJ} \approx 16.2 \text{kJ} \]
Problem 22.33

In a cylinder of an automobile engine just after combustion, the gas is confined to a volume of 50.0 cm$^3$ and has an initial pressure of $3\times10^6$ Pa. The piston moves outward to a final volume of 300 cm$^3$, and the gas expands without energy loss by heat. (a) If $\gamma = 1.4$ for the gas, what is the final pressure? How much work is done by the gas in expanding?

With good approximation, we can assume that the gas in the engine is ideal which satisfies the following state equation (relating pressure $P$, volume $V$ and the temperature $T$ of the gas)

1) \[ PV = nRT \]

where $n$ is the amount of the gas expressed in moles and $R$ is the gas constant. For an ideal gas, the change in internal energy is related to the change in temperature only

2) \[ dU = nC_v dT \]

where $C_v$ is the molar heat capacity at constant volume.

According to the first law of thermodynamics, for an adiabatic process (no heat delivered to the gas), the change in internal energy is opposite to the work performed by the gas

3) \[ dU = 0 - PdV \]

The rest is math. Since the pressure and volume is given in the problem we want to eliminate the temperature from the consideration. From equation (1), we can express the temperature differential in terms of the volume and pressure differentials

\[ PdV + VdP = nRdT \]

Using (2) and (3) we can eliminate temperature from last equation

\[ PdV + VdP = nR \cdot \frac{-PdV}{nC_v} \]

Because the difference between the molar heat capacity as the constant pressure and the constant volume for one mole of gas is equal to the gas constant
4) \[ C_p - C_v = R \]

we can rearrange the last equation and write

\[ P \left( 1 + \frac{C_p - C_v}{C_v} \right) dV = -V dP \]

or simplify it even further by substituting the ratio of the molar heat capacity at constant pressure and the molar heat capacity at constant volume

5) \[ \frac{C_p}{C_v} = \gamma \]

Also dividing both sides of the equation by the product of volume and pressure we can separate the variables (group terms with volume on one side of the equation and terms with pressure on the other side of the equation)

\[ \gamma \frac{dV}{V} = -\frac{dP}{P} \]

Integrating both side over the considered process (from initial volume and pressure to the final volume and pressure)

\[ \int_{\text{process}} \gamma \frac{dV}{V} = \int_{\text{process}} -\frac{dP}{P} \]

we get

\[ \gamma (\ln V_f - \ln V_i) = -(\ln P_f - \ln P_i) \]

Using the properties of logarithmic function we can simplify the last equation

\[ \ln \left( \frac{V_f}{V_i} \right)^\gamma = -\ln \frac{P_f}{P_i} \]

Therefore

\[ \frac{V_i^\gamma}{V_f^\gamma} = \frac{P_f}{P_i} \]
(We can present the last equation in the traditionally used form
\[ P_f V_f^\gamma = P_i V_i^\gamma \])

Solving for the only unknown (the final pressure) we can find the answer
\[
P_f = P_i \frac{V_i^\gamma}{V_f^\gamma} = 3 \cdot 10^6 \text{Pa} \cdot \left(\frac{50 \text{cm}^3}{300 \text{cm}^3}\right)^{1.4} = 1.2 \cdot 10^5 \text{Pa}
\]

Comment. In this problem it is assumed that nitrogen undergoes the process. In the combustion, significant amounts of water (H\(_2\)O) and carbon dioxide (CO\(_2\)) are produced. Therefore, it would be more accurate to assume that 1.3<\(\gamma\)<1.4. (See table 21.2)
Problem 22.41

A 2-L container has a center partition that divides it into two equal parts, as shown in Figure P22.41. The left-hand side contains H₂ gas, and the right-hand side contains O₂ gas. Both gases are at room temperature and at atmospheric pressure. The partition is removed, and the gases are allowed to mix. What is the entropy increase of the system?

a) When the gases are mixed they fill the entire volume of the container. The initial and the final temperatures are equal, therefore we can choose an isothermal process in order to calculate the change in entropy. (The isobaric process is incorrect, because although the pressure in the mixture is 1 atm as each gas expands and its partial pressure decreases.)

In the considered conditions, both gases can be treated as ideal gases. Their partial pressure P, volume V and temperature T are related by the equation of the state of an ideal gas.

1) \( PV = nRT \)

In the isothermal process for an ideal gas, the internal energy of the gas does not change. From the first law of thermodynamics, the heat delivered to the ideal gas must be equal to the work done by the gas in an isothermal process.

2) \( dQ_r = dW_r = PdV = \frac{nRTdV}{V} \)

Using the definition of entropy, the change in entropy of the ideal gas undergoing an infinitesimal expansion is therefore

\[ dS = \frac{dQ_r}{T} = \frac{nRdV}{V} \]

The entropy in the entire expansion process of an ideal gas changes by
\[ \Delta S_x = \int_{V_{xi}}^{V_{xf}} \frac{nRdV}{V} = nR \ln \frac{V_{xf}}{V_{xi}} \]

Both the hydrogen and the oxygen increase entropy. The change in the entropy of the system is therefore

\[ \Delta S_{H_2} + \Delta S_{O_2} = n_{H_2} R \ln \frac{V_{H_2,f}}{V_{H_2,i}} + n_{O_2} R \ln \frac{V_{O_2,f}}{V_{O_2,i}} = \]

\[ = 0.44 \text{mol} \cdot 8.31 \frac{\text{J}}{\text{mol} \cdot \text{K}} \cdot \ln \frac{2 \text{l}}{1 \text{l}} + 0.44 \text{mol} \cdot 8.31 \frac{\text{J}}{\text{mol} \cdot \text{K}} \cdot \ln \frac{2 \text{l}}{1 \text{l}} = 0.507 \frac{\text{J}}{\text{K}} \]
Problem 22.62

One mole of an ideal monatomic gas is taken through the cycle shown in Figure P22.67. The process A→B is a reversible isothermal expansion. Calculate (a) the net work done by the gas, (b) the (thermal) energy added to the gas, (c) the (thermal) energy expelled by the gas, and (d) the efficiency of the cycle.

![Diagram of gas cycle with points A, B, and C on a pressure-volume graph.]

a) In each process the work depends on the change in volume of the gas and the pressure of the gas during the process.

1) \[ dW = PdV \]

In order to find the value of the work, we have to know the explicit dependence of the pressure during the process.

In an isothermal process AB, the pressure \( P \) of the gas is inversely proportional to its volume \( V \)

2) \[ P(V) = \frac{nRT}{V} \]

Therefore the work done in the isothermal process is

\[
W_{AB} = \int_{V_A}^{V_B} PdV = \int_{V_A}^{V_B} \frac{nRT}{V}dV = P_AV_A \ln \frac{V_B}{V_A} = \\
= 5\text{atm} \cdot \frac{1.013 \cdot 10^5 \text{Pa}}{1\text{atm}} \cdot 10 \cdot 10^{-3} \text{m}^3 \ln \frac{50\text{l}}{10\text{l}} = 8151.8\text{J}
\]
In process BC, the pressure of the gas is constant. In this isobaric process the integration will be much easier

\[ W_{BC} = \int_{V_B}^{V_C} PdV = P_B (V_C - V_A) = 
= 1.013 \cdot 10^5 \text{Pa} \cdot (10 \cdot 10^{-3} \text{m}^3 - 50 \cdot 10^{-3} \text{m}^3) = -4052 \text{J} \]

In the third process, the volume is not changed, therefore the gas does not perform work.

The work done by the gas in the whole cycle is therefore

\[ \Delta W = W_{AB} + W_{BC} + W_{CA} = 8152 \text{J} - 4052 \text{J} + 0 \text{J} = 4 \text{kJ} \]

b,c) We have to perform similar calculations for the heat. In the isothermal expansion of an ideal gas its internal energy does not change. According to the first law of thermodynamics, the heat delivered to the gas is equal to the work done by the gas in this process.

3) \[ Q_{AB} = \Delta U_{AB} + W_{AB} = W_{AB} = 8.1 \text{kJ} \]

In this process the heat is delivered to the gas.

In the isobaric process the temperature of the gas changes. We can express the heat \( Q_{BC} \) delivered to the gas, with the change in its temperature and the molar heat capacity \( C_p = \frac{5}{2} R \) (at constant pressure) of a monatomic ideal gas\(^1\)

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\(^1\) See chapter 21 for the molar heat capacity of an ideal gas.
In this process the gas expels the heat.

In the isochoric process the temperature of the gas also changes. We can express the heat delivered to the gas, in terms of the change in its temperature and the molar heat capacity $C_v = \frac{3}{2}R$ (at constant volume) of a monatomic ideal gas

$$Q_{CA} = \int_{T_c}^{T_A} nC_v \, dT = nC_v(T_A - T_B) = \frac{3}{2} nR \left( \frac{P_A V_A}{nR} - \frac{P_C V_C}{nR} \right) = \frac{3}{2} (P_A V_A - P_C V_C) =$$

$$= \frac{3}{2} \left( 5 \text{atm} \cdot \frac{1.013 \cdot 10^5 \text{Pa}}{\text{latm}} \cdot 10 \cdot 10^{-3} \text{m}^3 - 1.013 \cdot 10^5 \text{Pa} \cdot 50 \cdot 10^{-3} \text{m}^3 \right) = 6.1 \text{kJ}$$

Heat is delivered to the gas in this process.

In this cycle heat was delivered to the gas from the heat reservoir in processes $AB$ and $CA$. The heat delivered in the whole cycle is therefore

$$\Delta Q_h = Q_{AB} + Q_{CA} = 8.1 \text{kJ} + 6.1 \text{kJ} = 14.2 \text{kJ}$$

In this cycle, the gas only in process $BC$ expels heat. The heat expelled in the whole cycle is therefore

$$\Delta Q_c = -Q_{BC} = 10.1 \text{kJ}$$

(To verify our calculations we can check if energy is conserved. In a cycle the internal energy of the gas does not change therefore the energy delivered must be equal to the sum of the energy used and wasted in the heat sink

$$14.2 \text{kJ} = 4.1 \text{kJ} + 10.1 \text{kJ}.$$
In the solution, we could use this fact to find heat delivered to the gas in one of the three processes.

d) From the definition of efficiency, we have to compare the energy used in the form of work performed by the gas to the energy delivered to the gas engine from the reservoir

\[ e = \frac{\Delta W}{\Delta Q_h} = \frac{4.1 \text{kJ}}{14.2 \text{kJ}} = 28.9\% \]