Test 1
phy 240

1. a) What is the purpose of measurement?
   b) Write all four conditions, which must be satisfied by a scalar product. (Use different symbols to distinguish operations on vectors from operations on numbers.)
   c) What is a projection of a vector?
   d) What does it mean that multiplication of a vector by a number is distributive over addition of numbers? (Use unambiguous symbols to distinguish operations on vectors from operations on numbers.)

2. In a Cartesian coordinate system, vectors $\mathbf{A}$ and $\mathbf{B}$ are given by the following linear combinations of the base vectors: $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$.
   a) Assign (isomorphically) a pair of scalar components to each vector.
   b) Find the scalar product of the two vectors.
   c) Determine the angle between the vectors.

3. What mass of a material with density $\rho$ is required to make a hollow spherical shell having inner radius $r_i$ and outer radius $r_o$?

4. Two vectors $\mathbf{A}$ and $\mathbf{B}$ have precisely equal magnitudes. For the magnitude of $\mathbf{A} + \mathbf{B}$ to be 100 times greater than the magnitude of $\mathbf{A} - \mathbf{B}$ what must the angle between them be? (Hint: $\sin^2 \frac{\theta}{2} = \frac{1}{2} (1 - \cos \theta)$; $\cos^2 \frac{\theta}{2} = \frac{1}{2} (1 + \cos \theta)$)
a) In a measurement, a mathematical quantity (number, vector, ...) is assigned to a physical quantity.

b) The following conditions are satisfied by scalar product:

1. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ (commutative)
2. $(\alpha \cdot \mathbf{A}) \cdot \mathbf{B} = \alpha (\mathbf{A} \cdot \mathbf{B})$ (mixed associative)
3. $(\mathbf{A} \oplus \mathbf{B}) \cdot \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}) + (\mathbf{B} \cdot \mathbf{C})$ (distributive over addition of vectors)
4. $\mathbf{A}^2 \geq 0$
   \[ \mathbf{A}^2 = 0 \text{ if and only if } \mathbf{A} = \mathbf{0} \]

c) The projection $\mathbf{A}_i$, of vector $\mathbf{A}$, in the direction of a unit vector $\mathbf{e}_i$ is defined as
   \[ \mathbf{A}_i = (\mathbf{A} \cdot \mathbf{e}_i) \cdot \mathbf{e}_i \]

d) If $\alpha$ and $\beta$ are two arbitrary numbers, and $\mathbf{A}$ is an arbitrary vector then
   \[ (\alpha + \beta) \cdot \mathbf{A} = \alpha \cdot \mathbf{A} \oplus \beta \cdot \mathbf{A} \]
a) The coefficients in the linear combination of the base vectors constitute the scalar components of a vector. As both vectors are given in the form of the linear combination, the scalar components are practically given

\[
\vec{A} = [2,3] \quad \text{and} \quad \vec{B} = [-1,2]
\]

b) The scalar product of vectors \( \vec{A} \) and \( \vec{B} \) is equal to the scalar product of the \( \mathbb{R}^n \) vectors assigned in a Cartesian coordinate system

\[
\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{B} = [2,3] \cdot [-1,2] = -2 + 6 = 4
\]

c) Following the definition of the angle between the vectors, the angle between vectors \( \vec{A} \) and \( \vec{B} \) is

\[
\phi = \arccos \frac{\vec{A} \cdot \vec{B}}{\| \vec{A} \| \| \vec{B} \|} = \arccos \frac{[2,3] \cdot [-1,2]}{\sqrt{[2,3]^2} \cdot \sqrt{[-1,2]^2}} = \arccos \frac{4}{\sqrt{13} \cdot \sqrt{5}} \approx 60^\circ
\]
(From the definition of density, we can relate the mass \( dm \) of a “small” (infinitesimal) segment to the appropriate (differential) volume of the segment:

\[
(1) \quad dm = \rho dV
\]

To find the mass of the entire object, we have to integrate (“add”) the mass of all the segments. “Adding” the mass is based on the scalar property of this quantity.)

Since the density is uniform over the volume of the body the integration is significantly simplified.

\[
(2) \quad m = \int_{\text{shell}} \rho dV = \rho \int_{\text{shell}} dV = \rho V
\]

Knowing the radii, we can find the volume of the shell directly from the definition of volume, calculating the appropriate integral. It is easier though to use the scalar property of volume and the generally known formula for the volume of a sphere (theorem). The volume of the shell is equal to the difference in volume of the outer and the inner sphere:

\[
(3) \quad V = \frac{4}{3} \pi r_0^3 - \frac{4}{3} \pi r_i^3
\]

Equations (2) and (3) form a set with two unknowns \( \{m, V\} \). Solving (by substitution) we find the mass of the shell (the answer):

\[
m = \rho V = \frac{4}{3} \pi (r_0^3 - r_i^3) \rho
\]
Solution 1

Let $\theta$ represent the angle between the directions of the two vectors. From the definition of magnitude

$$|A + B| = \sqrt{(A + B)^2} = \sqrt{A^2 + B^2 + 2A \cdot B} = \sqrt{2A^2(1 + \cos \theta)} = 2A \cos \frac{\theta}{2}$$

$$|A - B| = \sqrt{(A - B)^2} = \sqrt{A^2 + B^2 - 2A \cdot B} = \sqrt{2A^2(1 - \cos \theta)} = 2A \sin \frac{\theta}{2}$$

The ratio given in the problem yields the following trigonometric equation

$$\frac{1}{100} = \frac{|A - B|}{|A + B|} = \frac{2A \cos \frac{\theta}{2}}{2A \sin \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

from which

$$\theta = 2 \cdot \arctan \frac{1}{100} \approx 0.02 \approx 1.15°$$
Solution 2 (for oriented segments)

Since $\vec{A}$ and $\vec{B}$ have the same magnitudes (lengths), $\vec{A}$, $\vec{B}$, and $\vec{A} + \vec{B}$ form an isosceles triangle in which the angles are $(\pi - \theta)$, $\theta/2$ and $\theta/2$. Similarly $\vec{A}$, $\vec{B}$, and $\vec{A} - \vec{B}$ form an isosceles triangle in which the angles are $\theta$, $(\pi - \theta)/2$ and $(\pi - \theta)/2$. From the law of cosines, the magnitude of both $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$ can be expressed in terms of the magnitude of vector $\vec{A}$ (and $\vec{B}$)

\[
|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 - 2AB \cos(\pi - \theta)} = 2A \sin \frac{\theta}{2}
\]
\[
|\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB \cos \theta} = 2A \sin \frac{\theta}{2}
\]