Homework 10

chapter 30: 14, 17, 25, 41
**Problem 30.14**

Determine the magnetic field (in terms of I, a, and d) at the origin due to the current loop in Figure P30.14.

Using the right hand rule we should see that the vertical pieces of wire produce magnetic field in the direction of $\hat{k}$ unit vector (positive z-axis) while the horizontal piece of wire produces a field in $-\hat{k}$ direction (negative z-axis). Finding the magnetic field in this problem requires integration of the equation given in the Biot-Savart law. It is convenient to separate the integral into three parts: integration along piece (i), integration along piece (ii), and integration along piece (iii). (It is difficult to introduce a single integration variable to identify each differential segment along the wire.)

From geometrical considerations, the integrated function (in the Biot-Savart law) for each piece can expressed in terms of the integration variable. For the first piece (i) I will choose $y$ as the integration variable in the limits from $d$ to $\infty$,

$$\mathrm{dB}' = \frac{\mu_0}{4\pi} \cdot \frac{\mathrm{ds}' \times \hat{r}'}{r'^2} = \frac{\mu_0 I \hat{k}}{4\pi} \cdot \frac{\sin \theta'}{r'^2} \cdot \mathrm{d}y = \frac{\mu_0 I \hat{k}}{4\pi} \cdot \frac{1}{a^2 + y^2} \cdot \frac{a}{\sqrt{a^2 + y^2}} \cdot \mathrm{d}y$$

Therefore

$$\mathbf{B}' = \frac{\mu_0 I a \hat{k}}{4\pi} \cdot \int_d^{\infty} \frac{1}{\left( a^2 + y^2 \right)^{\frac{3}{2}}} \cdot \mathrm{d}y = \frac{\mu_0 I a \hat{k}}{4\pi a^2} \cdot \frac{y}{\sqrt{a^2 + y^2}} \bigg|_{d}^{\infty} = \frac{\mu_0 I}{4\pi a} \cdot \left( 1 - \frac{d}{\sqrt{a^2 + d^2}} \right) \cdot \hat{k}$$
Using geometrical considerations, we can relate the differential magnetic field produced by the horizontal piece of wire with another convenient integration variable. I decided to use \( x \) to identify each segment and therefore I expressed all functions in terms of this variable. From the Biot-Savart law

\[
\frac{dB''}{4\pi} = \frac{I ds \times \hat{r}}{r^2} = -\frac{\mu_0}{4\pi} \frac{I \sin \theta dx \hat{k}}{r^2} = -\frac{\mu_0 I \hat{k}}{4\pi} \frac{d}{(d^2 + x^2)^{3/2}} dx
\]

Now we have to integrate over the horizontal segment, which specifies the limit of integration to be \(-a\) and \(a\).

\[
B'' = -\frac{\mu_0 I d}{4\pi} \int_{-a}^{a} \frac{1}{(d^2 + x^2)^{3/2}} dx = -\frac{\mu_0 I}{2\pi d} \frac{a}{\sqrt{d^2 + a^2}} \cdot \hat{k}
\]

The symmetry of the arrangement indicates that both vertical pieces of wire produce identical magnetic field at the origin. The resulting magnetic field is therefore

\[
B = 2B' + B'' = \frac{\mu_0 I}{2\pi} \left[ \frac{1}{a} \left( 1 - \frac{d}{\sqrt{a^2 + d^2}} \right) - \frac{a}{d \sqrt{d^2 + a^2}} \right] \cdot \hat{k} = -\frac{\mu_0 I}{2\pi ad} \frac{a^2 + d^2 - d \sqrt{a^2 + d^2}}{\sqrt{a^2 + d^2}} \cdot \hat{k}
\]

The number in front of the unit vector \( \hat{k} \) is always negative. (The resulting magnetic field is into the page.)
Problem 30.17

The current in the long, straight wire is $I_1 = 5 \text{ A}$ and the wire lies in the plane of the rectangular loop, which carries current $I_2 = 10 \text{ A}$. The dimensions are $c = 0.1 \text{ m}$, $a = 0.15 \text{ m}$, and $l = 0.45 \text{ m}$. Find the magnitude and direction of the net force exerted on the loop by the magnetic field created by the wire.

a) The long wire produces a concentric magnetic field. Using the right-hand rule one can determine that the magnetic field vector is perpendicular to the plane of the loop as marked in the figure. From symmetry one can also predict that the force exerted on the two sides of the loop which are perpendicular to the long wire are opposite. The magnetic force exerted on the entire loop result only from the interaction of the sides parallel to the long wire.

Again using right hand rule one can predict that the direction of the force exerted on the side closer to the long wire is attractive and the force exerted on the farther side is repulsive. Since the magnetic field is inversely proportional to the distance between the interacting wires, the force exerted on the closer side of the loop is stronger than the force exerted on the farther side of the loop. Hence the magnetic force exerted on the entire wire is toward the long wire and the magnitude of the magnetic force is
\[ F_B = I_2 l \cdot \left( \frac{\mu_0 I_1}{2\pi c} \right) \cdot \sin 90^\circ - I_2 l \cdot \left( \frac{\mu_0 I_1}{2\pi (c + a)} \right) \cdot \sin 90^\circ = \frac{\mu_0 I_1 I_2 l}{2\pi} \cdot \left( \frac{1}{a} - \frac{1}{c + a} \right) = \]
\[ = \left( 4\pi \cdot 10^{-7} \text{ N/A}^2 \right) \cdot 5\text{ A} \cdot 10\text{ A} \cdot 0.45\text{ m} \cdot \left( \frac{1}{0.1\text{ m}} - \frac{1}{0.1\text{ m} + 0.15\text{ m}} \right) = 2.7 \cdot 10^{-5} \text{ N} \]
Problem 30.25

The figure shows a cross-section view of a coaxial cable. The center conductor is surrounded by a rubber layer, which is surrounded by an outer conductor, which is surrounded by another rubber layer. In a particular application, the current in the inner conductor is 1 A out of the page and the current in the outer conductor is 3 A into the page. Determine the magnitude and direction of the magnetic field at points a and b.

The system has axial symmetry and therefore the electric field can be found from the Ampere-Maxwell equation. Considering closed loops, concentric with the cable and passing by the points of interest, the line integral can be related to the radius of each loop and the current flowing through the surface spanned by the loop.

At point A

\[ B \cdot 2\pi r_A = \oint B \cdot ds \cos 0^\circ = \oint B \cdot ds = \mu_0 I_A \]

from which

\[ B = \frac{\mu_0 I_A}{2\pi r_A} = \frac{4\pi \cdot 10^{-7} \text{ N} / \text{A}^2 \cdot 1 \text{A}}{2\pi \cdot 0.01 \text{m}} = 2 \cdot 10^{-5} \text{ T} \]

in the “positive” y-direction
At point B
\[ B \cdot 2\pi r_A = \oint B \cdot ds = \oint B \cdot ds = \mu_0 (-I_B) \]
from which
\[ B = -\frac{\mu_0 I_B}{2\pi r_A} = -\frac{4\pi \cdot 10^{-7} \text{ N} / \text{A}^2 \cdot (-2A)}{2\pi \cdot 0.03 \text{m}} = 1.3 \cdot 10^{-5} \text{T} \]
in the “negative” y-direction
Problem 30.41

A solenoid 2.5 cm in diameter and 30 cm long has 300 turns and carries 12 A. (a) Calculate the flux through the surface of the disk of radius 5.0 cm that is positioned perpendicular to and centered on the axis of the solenoid, as in Figure†P30.34a. (b) Figure P30.34b shows an enlarged and view of the solenoid described above. Calculate the flux through the blue area, which is defined by an annulus that has an inner radius of 0.40 cm and outer radius of 0.80 cm.

(a) From the definition of the flux of a vector, the magnetic flux (the flux of the magnetic field vector) through the surface (disc) is

\[ \Phi_B = \int_{\text{disk}} B \cdot dA \]

(I calculate the flux in the direction marked in the figure.)

The magnetic field is uniform inside the solenoid. Its direction coincides with the direction of the solenoid's axis and its magnitude depends on the current in the wire, and the number of turns per unit length of the solenoid.

\[ B = \mu_0 \frac{N}{L} I \]

The magnetic field outside the solenoid is a zero vector.
We can split the integration over the area of the disk into two parts: the part ("in") inside the solenoid and the part ("out") outside the solenoid. Therefore

\[
\Phi_B = \int_{\text{in}} \mathbf{B} \cdot d\mathbf{A} + \int_{\text{out}} \mathbf{B} \cdot d\mathbf{A} = \mu_0 \frac{N}{L} I \cdot \pi r^2 + 0 \cdot \pi R^2 =
\]

\[
= 4\pi \cdot 10^{-7} \frac{N}{A^2} \cdot \frac{300}{0.3m} \cdot 12A \cdot \pi \left(\frac{2.5}{2} \cdot 10^{-2} m\right)^2 = 7.4 \cdot 10^{-6} \text{Wb}
\]

Note.

\[
\frac{N}{A^2 \cdot m} \cdot A \cdot m^2 = \frac{N}{A \cdot m} \cdot m^2 = T \cdot m^2 = \text{Wb}
\]

b) The shaded surface is entirely inside the solenoid therefore the magnetic field has the same value at all its points. It simplifies the integration of the flux

\[
\Phi'_B = \int_{\text{surface}} \mathbf{B} \cdot d\mathbf{A} = B \cdot \int_{\text{surface}} d\mathbf{A} \cdot \cos 90^\circ = \mu_0 \frac{N}{L} I \cdot \pi (r_{\text{outer}}^2 - r_{\text{inner}}^2) =
\]

\[
= 4\pi \cdot 10^{-7} \frac{N}{A^2} \cdot \frac{300}{0.3m} \cdot 12A \cdot \pi \left((0.8 \cdot 10^{-2} m)^2 - (0.4 \cdot 10^{-2} m)^2\right) = 2.27 \cdot 10^{-6} \text{Wb}
\]