Homework 11

chapter 31: 15, 27, 35, 61
Problem 31.15

A long solenoid has \( n = 400 \) turns per meter and curries current given by \( I = 30A \cdot (1-e^{-1.6t}) \). Inside the solenoid and coaxial with is a coil that has a radius of 6 cm and consists of a total of \( N = 250 \) turns of fine wire. What emf is induced in the coil by the changing current?

A long solenoid produces a uniform magnetic field. The strength \( B \) of the field depends on the number of turns per unit length \( n \) and the current in the solenoid \( I \)

\[
B = \mu_0 n I
\]

where \( \mu_0 \) is the permeability of free space. Therefore the flux through the surface spanned by the coil (in the direction of the field vector) is equal to the product of the magnetic field strength and the area of the coil

\[
\Phi_B = \oint B \cdot dA = \oint B \cdot dA \cdot \cos 0 = BA = \mu_0 n I \cdot \pi r^2
\]

From the Faraday’s law of induction, the indicated electromotive force induced in the coil

\[
\varepsilon = -N \frac{d\Phi_B}{dt} = -N \cdot \mu_0 n \cdot \pi r^2 \cdot \frac{dI}{dt} = -\mu_0 N n \pi r^2 \cdot \frac{d}{dt} I_0 e^{-\alpha t} = \mu_0 N n \pi r^2 I_0 \alpha e^{-\alpha t} = \left(4\pi \cdot 10^{-7} \frac{N}{A^2}\right) \cdot 250 \cdot 400 \cdot \pi \cdot (0.06m)^2 \cdot 30A \cdot 1.6 \frac{1}{s} \cdot e^{-1.6t} = 0.068V \cdot e^{-1.6t}
\]

Consistently with the figure, a positive value of the current indicates that the electromotive force in the coil is in the same direction as the current flowing in the solenoid.
**Problem 31.27**

A helicopter has blades 3 m long, extending from the central hub and rotating at 2 rev/s. If the vertical component of the earth magnetic field is 50 mT, what is the emf induced between the blade tip and the center hub?

A blade, rotating with magnitude of angular velocity \( \omega \) sweeps an angle \( d\theta \) proportional to the time \( dt \)

\[ d\alpha = \omega dt \]

which corresponds to an area of

\[ dA = \frac{1}{2} \cdot R d\alpha \cdot R = \frac{1}{2} R^2 \omega dt. \]

In a uniform magnetic field, the magnetic flux swept by the blade is therefore

\[ d\Phi_B = \mathbf{B} \cdot d\mathbf{A} = B_v dA = \frac{1}{2} B_v R^2 \omega dt \]

Hence the electromotive force induced in the blade is

\[ \varepsilon = -\frac{d\Phi_B}{dt} = \frac{1}{2} B_v R^2 \omega = \frac{1}{2} \cdot 50 \mu T \cdot (3 m)^2 \cdot 4\pi \frac{1}{s} = 2.8 \text{ mV} \]
Problem 31.35

A loop of area 0.1 m² is rotating at 60 rev/s with the axis of rotation perpendicular to a 0.2 T magnetic field. (a) If there are 1000 turns on the loop, what is the maximum voltage induced in the loop? (b) When the maximum induced voltage occurs, what is the orientation of the loop with respect to the magnetic field?

As the loop rotates, the angle between the direction of the magnetic field vector \( \mathbf{B} \) and the vector \( \mathbf{A} \), normal to the loop with magnitude equal to the area of the loop, is a linear function of time.

The magnetic flux \( \Phi \) over the surface of the loop is therefore a time dependent function, and at instant \( t \) its value is:

\[
\Phi(t) = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{A} = \mathbf{B} \cdot \mathbf{A} = BA \cos \omega t
\]

The electromotive force in the loop depends on the number of loops and the rate of change in the magnetic flux over the loop (or any other surface bound to the loop):

\[
\varepsilon(t) = -N \frac{d\Phi}{dt} = -N \frac{d}{dt}(BA \cos \omega t) = NAB\omega \sin \omega t
\]

The rest of the problem involves an analysis of the above function
a) There is no mention of polarity, therefore the maximum voltage (electromotive force) occurs for the extreme values of the trigonometric function. The absolute value of this voltage is:

\[ V_{\text{max}} = N A B \omega = 1000 \cdot 0.1 \text{m}^2 \cdot 0.2 \text{T} \cdot 2\pi \cdot 60 \text{s}^{-1} = 7540 \text{V} \]

b) To answer this part we must analyze and interpret for what values of the argument the function assumes the extreme values. It happens when the argument of the trigonometric function is

\[ \omega t = \frac{\pi}{2} + n\pi \]

Recall that \( \omega t \) represents the angle between the magnetic field and the normal to the loop. Therefore, the voltage assumes maximum values when the magnetic field is parallel to the plane of the loop.
Problem 31.61

A conducting rod of length $l$ moves with velocity $v$ along a direction parallel to a long wire carrying a steady current $I$. The axis of the rod is maintained perpendicular to the wire with the near end a distance $r$ away, as shown in Figure P31.63. Show that the emf induced in the rod is given by

$$|\mathcal{E}| = \frac{\mu_0 I}{2\pi} v \ln\left(1 + \frac{l}{r}\right)$$

Solution 1.

One way to solve this problem could be to find the electric field vector inside the rod from the equilibrium condition. The Lorentz force exerted on the electric particles inside the rod must be zero. Recalling the expression for the magnetic field vector of an infinite wire, the electric field $E(y)$ at location $y$ from the end of the rod closer to the wire must satisfy:

$$E(y) = v B(y) \sin 90^\circ = v \cdot \frac{\mu_0 I}{2\pi (r + y)}$$

(The direction of the electric field is away from the wire.)
In order to find the absolute value of the voltage between the ends of the rod, we have to recall the relationship between the electric field vector and the electric potential:

\[ V = V_+ - V_- = \int_0^\ell E \cdot ds = \int_0^\ell \frac{\mu_0 I}{2\pi} \cdot \frac{1}{r+y} \, dy = \frac{\mu_0 I}{2\pi} (r+y) \ln(1+\ell) \]

\[ = \frac{\mu_0 I}{2\pi} \cdot v \cdot \ln\left(1 + \frac{\ell}{r}\right) \]

Solution 2.

We can consider an arbitrary amperian loop (marked with a broken line) and use Faraday's law of induction. The electromotive force in this loop is created only in the rod.

At instant \( t \), the magnetic flux over the loop is

\[ \Phi(t) = \oint_{\text{loop}} B \cdot dA = \int_0^\ell \int_0^{x(t)} \frac{\mu_0}{2\pi (r+y)} \cdot \cos 0 \cdot dx \, dy = \frac{\mu_0}{2\pi} x(t) \cdot \ln\left(1 + \frac{\ell}{r}\right) \]

According to Faraday's law we have to find the rate of change of the above function:

\[ |\mathcal{E}| = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\mu_0}{2\pi} \ln\left(1 + \frac{\ell}{r}\right) \cdot x(t) \right) = \frac{\mu_0}{2\pi} \ln\left(1 + \frac{\ell}{r}\right) \cdot v \]