Homework 2

chapter 24: 18,

chapter 25: 19, 31, 35
Problem 24.18

A solid sphere of radius 40 cm has a total positive charge of 26 μC uniformly distributed throughout its volume. Calculate the magnitude of electric field (a) 0 cm, (b) 10 cm, (c) 40 cm, and (d) 60 cm from the center of the sphere.

Let's first find the electric field vector as a function of location in the general form. The spherical symmetry requires that the electric field vector at each location have a radial direction from the center of the sphere. That symmetry also requires that the magnitude of the electric field have the same value along a spherical surface concentric with the charged sphere.

From the definition of volume charge density and the fact that the density in this case is uniform, we can determine that charge density in our problem:

\[ \rho(r) = \frac{dq}{dV} = \frac{Q}{V} = \frac{3Q}{4\pi R^3} \quad (for \ r < R) \]
Now we can use Gauss' law to relate the magnitude of electric field with the distance \( r \) from the center of the charged sphere. Let's choose a spherical Gaussian surface concentric with the charged sphere. The angle between the electric field vector and the normal to the surface in this case is 0. Therefore, the flux through the Gaussian surface expressed in terms of the magnitude of the electric field at the Gaussian surface is

\[
(2) \quad \Phi = \oint \mathbf{E} \cdot d\mathbf{A} = \int \mathbf{E} \cdot \mathbf{A} = 4\pi r^2 E
\]

From Gauss' law, we know that that flux is proportional to the electric charge inside the Gaussian surface.

\[
(3) \quad Q(r) = \frac{4}{3} \pi r^3 \rho = \frac{Q}{R^3} r^3 \quad \text{for} \quad r < R
\]

\[
Q(r) = Q \quad \text{for} \quad r \geq R
\]

Therefore

\[
4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \cdot \frac{Q}{R^3} r^3 \quad \text{for} \quad r < R
\]

\[
4\pi r^2 E(r) = \frac{1}{\varepsilon_0} Q \quad \text{for} \quad r \geq R
\]

Hence

\[
E(r) = \begin{cases} 
\frac{kQ}{R^3} \cdot r & \text{for} \quad r \leq R \\
\frac{kQ}{r^2} & \text{for} \quad r \geq R 
\end{cases}
\]
Now we can use the above general expression to find electric field strength at the points indicated in the problem.

\[
\begin{align*}
a) \quad E(0\text{cm}) &= \frac{kQ}{R^3} \cdot 0m = 0 \frac{N}{C} \\
\quad &= \frac{9 \cdot 10^9 \frac{Nm^2}{C^2} \cdot 26 \cdot 10^{-6} C}{(0.4m)^3} \cdot 0.1m = 3.66 \cdot 10^5 \frac{N}{C} \\
b) \quad E(10\text{cm}) &= \frac{9 \cdot 10^9 \frac{Nm^2}{C^2} \cdot 26 \cdot 10^{-6} C}{(0.4m)^3} \cdot 0.1m = 3.66 \cdot 10^5 \frac{N}{C} \\
c) \quad E(40\text{cm}) &= \frac{9 \cdot 10^9 \frac{Nm^2}{C^2} \cdot 26 \cdot 10^{-6} C}{(0.4m)^2} = 1.46 \cdot 10^6 \frac{N}{C} \\
d) \quad E(40\text{cm}) &= \frac{9 \cdot 10^9 \frac{Nm^2}{C^2} \cdot 26 \cdot 10^{-6} C}{(0.6m)^2} = 6.5 \cdot 10^5 \frac{N}{C}
\end{align*}
\]

(Note. There is no question about the direction in the problem.)
Problem 25.19

Show that the amount of work required to assemble four identical point charges of magnitude Q at the corners of a square of side s is 5.41 kQ^2/s.

The question refers to the work performed by “external” object, other than the four charged particles. The particles also perform “internal” work on each other.

We can assume that the particles are initially far separated and have no initial kinetic energy. The kinetic energy is also zero when the system is assembled. Since electrostatic interaction is conservative it does not matter how the system is assembled. We can assume that the particles are brought to their final positions one by one. In order to find the answer, we can add the “external” work performed on each particle.

\[ \Delta W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \Delta W_4 \]

We can use the work-energy theorem to find each work. For each particle (i) the “external” work is equal to the change in its mechanical energy

\[ \Delta W_i = \Delta K_i + \Delta U_i \]

where K is the kinetic energy and U is the potential energy due to electrostatic interaction. (From the initial assumption about the kinetic energy we see that the change in the kinetic energy is zero.)
The first particle is completely free to move (no other particle interact with it) therefore it requires no work to place it at the right position

$$\Delta W_1 = 0$$

The second particle has a nonzero potential energy due to the first particle. Moving the second particle is associated with a change in its potential energy $U_2$ (due to the potential $V_1$ produced by the first particle)

$$\Delta W_2 = U_{2f} - U_{2i} = Q(V_{1f} - V_{1i}) = \frac{kQ^2}{s}$$

Potential energy of the third particle results from interaction with the first two

$$\Delta W_3 = U_{3f} - U_{3i} = Q((V_{1f} + V_{2f}) - (V_{1i} + V_{2i})) = \frac{kQ^2}{s} \cdot \left(1 + \frac{\sqrt{2}}{2}\right)$$

Finally, three particles contribute to the potential energy of the fourth particle

$$\Delta W_4 = U_{4f} - U_{4i} = Q((V_{1f} + V_{2f} + V_{3f}) - (V_{1i} + V_{2i} + V_{3i})) = \frac{kQ^2}{s} \cdot \left(2 + \frac{\sqrt{2}}{2}\right)$$

Hence

$$\Delta W = \frac{kQ^2}{s} \cdot (4 + \sqrt{2}) \approx 5.41 \cdot \frac{kQ^2}{s}$$
Problem 25.31

Over a certain region of space, the electric potential is \( V = 5x - 3x^2y + 2yz^2 \). Find the expression for the x, y, and z components of the electric field over this region. What is the magnitude of the field at point P, which has coordinates \([1,0,-2]\) m?

In this problem, the electric potential is given at all points of the space, in terms of an explicit function of scalar components of the position vector. It should be mentioned that the coefficients in that function have such units so that the value of the function is in volts.

\[
V(x,y,z) = \left( \frac{5}{m} \right) x - \left( \frac{3}{m^3} \right) x^2 y + \left( \frac{2}{m} \right) y z^2
\]

For the sake of simplicity we can omit the units in the above equation but we have to remember that when the components of position are given in meters the value of potential is in volts.

The electric field vector is equal to the gradient of the electric potential for the considered electric field. Therefore the components of the electric field vector at location \( \mathbf{r} = [x,y,z] \) are:

\[
\mathbf{E}(\mathbf{r}) = -\nabla V = \left[ \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right] = \left[ 5 - 6xy, -3x^2 + 2z^2, 4yz \right]
\]

At point P, whose position is \( \mathbf{r}_p = [1,0,-2] \) m, the electric field vector has a value of

\[
\mathbf{E}(1\text{m,0m, }-2\text{m}) = -[5,5,0] \frac{\text{V}}{\text{m}}
\]

The magnitude of this vector (found from the definition) is
E(1m, 0m, -2m) = \sqrt{E_x^2 + E_y^2 + E_z^2} = 7.1 \frac{V}{m}

Note. Recall that when we introduced the concept of an electric field vector, we assigned N/C as the SI unit of this quantity. The units of the quantities discussed above are also SI units. We should make a comment about the unit we have here for the electric field vector.

\[ \frac{V}{m} = \frac{J}{Cm} = \frac{Nm}{Cm} = \frac{N}{C} \]
Problem 25.35

A rod of length L (Fig. P.25.43) lies along the x axis with its left end at the origin and has a nonuniform charge density \( \lambda = \alpha x \) (where \( \alpha \) is a positive constant).

(a) What are the units of \( \alpha \)? (b) Calculate the electric potential at A.

**a)** Since it is not otherwise specified, by default we should assume that the physical quantities discussed in the problem are expressed in SI units. Recall that the linear charge density is 1 C/m and the unit of length is 1 m. Whenever there is a relationship between two physical quantities expressed in the form of an equation, the units on both sides must be the same. This requires that the coefficient \( \alpha \) in the equation \( \lambda(x) = \alpha x \) has unit \( \text{C/m}^2 \).

**b)** When we divide the rod into differential segments, we can find the contribution to the electric potential \( dV_A \) from that segment, treating it like a point charge. We need the (differential) charge \( dq \) of the segment, related to the charge density at this location \( \lambda(x) \), and the distance of the considered location to the segment \( r'(x) = d + x \).

\[
dV_A = \frac{kdq}{r'} = \frac{k\alpha dx}{d + x}
\]

We obtain the total electric potential at point A, by integrating the above (adding) over the entire rod. Since we have chosen the x-component of position as the variable, consistent with the figure, the limits of integration are 0 and L.

\[
V_A = k\alpha \int_0^L \frac{dx}{d + x} = k\alpha \left[ \ln \left( \frac{x + d}{d} \right) \right]_0^L = k\alpha \left( L - d \ln \frac{L + d}{d} \right)
\]