Homework 6

chapter 32: 11, 18
chapter 33: 5, 50
Problem 32.11

A 12 V battery is connected into a series circuit containing a 10 Ω resistor and a 2 H inductor. In what time interval will the current reach (a) 50% and (b) 90% of its final value?

This circuit has only one loop. From Kirchhoff’s rule we get only one (differential) equation

\[ \varepsilon - L \frac{dI}{dt} - IR = 0 \]

We can separate the variables

\[ \frac{dI}{\varepsilon - IR} = \frac{dt}{L} \]

and integrate both sides of the equation (within appropriate limits). With the reference for time at the instant of closing the circuit

\[ \int_0^{I(t)} \frac{dI}{\varepsilon - IR} = \int_0^t \frac{dt}{L} \]

(Note that I used one symbol t for two different quantities!)

From the fundamental theorem of calculus

\[ \ln \frac{\varepsilon - I(t) \cdot R}{\varepsilon} = -\frac{R}{L} \cdot t \]

The solution is a time dependent function
\[ I(t) = \frac{\varepsilon}{R} \left( 1 - e^{-\frac{R}{L}t} \right) \]

a) Current reaches its final value after an infinite amount of time.

\[ I(\infty) = \lim_{t \to \infty} \frac{\varepsilon}{R} \left( 1 - e^{-\frac{R}{L}t} \right) = \frac{\varepsilon}{R} \]

At instant \( t \), the current is a fraction of its final value

\[ I(t_1) = I(\infty) \cdot \left( 1 - e^{-\frac{R}{L}t} \right) \]

From which

\[ \frac{I(t)}{I(\infty)} = \left( 1 - e^{-\frac{R}{L}t} \right) \]

Solving this equation for \( t \)

\[ t = -\frac{L}{R} \ln \left( 1 - \frac{I(t)}{I(\infty)} \right) \]

Hence

a) \( t_{50\%} = -\frac{2H}{10\Omega} \ln(1 - 0.5) = 0.14\)s

b) \( t_{90\%} = -\frac{2H}{10\Omega} \ln(1 - 0.9) = 0.46\)s
Problem 32.18

The switch in Fig. P32.18 is open for \( t < 0 \) then closed at time \( t = 0 \)s. Find the current in the inductor and the current in the switch as a function of time thereafter.

Using Kirchhoff’s rules we can write three independent equations. Consistent with the assumed direction of the three unknown currents. (For the sake of simplicity I will not write the units but I will make sure all quantities are in the SI system.)

\[
\begin{align*}
I_1 - I_2 &= I_3 \\
-4\Omega \cdot I_1 - 4\Omega \cdot I_2 &= -10V \\
4\Omega \cdot I_2 - 8\Omega \cdot I_3 - 1H \cdot \frac{dI_3}{dt} &= 0
\end{align*}
\]

The rest is math. From the first two equations we can find the substitution for \( I_2 \) in the last equation
\[
I_2 = \begin{vmatrix}
    1 & I_3 \\
   -4\Omega & -10V \\
   1 & 1 \\
   -4\Omega & -4\Omega \\
\end{vmatrix}
= -2 \cdot \begin{vmatrix}
    1 & I_3 \\
    2\Omega & 5V \\
   1 & 1 \\
   -4\Omega & -1 \\
\end{vmatrix}
= \frac{5V - 2\Omega \cdot I_3}{4\Omega}
\]

With this substitution only one variable (function) is left in the third equation

\[(5V - 2\Omega \cdot I_3) - 8\Omega \cdot I_3 - 1H \cdot \frac{dI_3}{dt} = 0\]

We can solve this differential equation separating the variables

\[
\frac{dI_3}{dt} = -10\Omega \cdot I_3 + 5V
\]

\[
\frac{dI_3}{-10\frac{1}{s} \cdot I_3 + 5\frac{A}{s}} = dt
\]

and integrating both sides (within appropriate limits)

\[
\int_{0A}^{I_3} \frac{dI_3}{-10\frac{1}{s} \cdot I_3 + 5\frac{A}{s}} = \int_{0s}^{t} dt
\]

(Do not confuse the symbol \( t \) in the limit with the integration variable \( t \)! I used one symbol for two different quantities.)

Using the fundamental theorem of calculus
\[
-\frac{1}{10}s \cdot \ln\left(\frac{-10\cdot I_3 + 5A}{s}\right) = t
\]

Hence (including the appropriate SI units)

\[
I_3 = 0.5A\left(1 - e^{-\frac{t}{0.1s}}\right)
\]

In order to find the current \(I_1\) in the switch we have to return to the first two equations

\[
I_1 = \begin{vmatrix}
I_3 & -1 \\
-10V & -4\Omega \\
1 & -1 \\
-4\Omega & -4\Omega
\end{vmatrix} = \begin{vmatrix}
I_3 & -1 \\
5V & 2\Omega \\
1 & -1 \\
4\Omega & 4\Omega
\end{vmatrix} = \frac{2\Omega \cdot I_3 + 5V}{4\Omega} = 1.25A + 0.25A\left(1 - e^{-\frac{t}{0.1s}}\right) = 1.5A - 0.25Ae^{-\frac{t}{0.1s}}
Problem 33.5

The current in the circuit shown in Figure 33.2 equals 60% of the peak value of the peak current at \( t = 7 \text{ ms} \). What is the lowest source frequency that gives this current?

The problem is phrased ambiguously. Since there is only one oscillating quantity, we can assume that the initial phase of the current is zero and the (instantaneous value of) current is a sinusoidal function of time

\[
I(t) = I_m \sin(2\pi f \cdot t)
\]

At the indicated instant \( t = 7 \text{ ms} \), the current assumes \( p = 60\% \) of its peak value. It requires that

\[
\sin(2\pi f \cdot t) = p
\]

This equation has an infinite number of solutions. The smallest frequency corresponds to the smallest argument of the sin function satisfying the equation

\[
2\pi f \cdot t = \arcsin p
\]

From which, the smallest frequency correspond is

\[
f_{\text{min}} = \frac{\arcsin p}{2\pi \cdot t} = \frac{\arcsin 0.6}{2\pi \cdot 7 \cdot 10^{-3} \text{s}} = 14.6\text{Hz}
\]
Problem 33.50

Show that the rms value for the sawtooth voltage shown in Figure P33.56 is $V_m/\sqrt{3}$.

The root-mean-square value of a function in a time interval $(t_1, t_2)$ is defined as

$$V_{\text{rms}}(t_1, t_2) = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} V^2(t) \, dt}$$

We can solve the problem directly from this definition.

We have to find the explicit form of the sawtooth function. Consistent with the symbols in the figure,

$$V(t) = \frac{2V_m}{T} \cdot t - (2N - 1)V_m \quad \text{for} \quad NT < t < (N+1)T$$

(In each time interval $[NT, (N+1)T]$, the voltage is a linear function of time and assumes the values $-V_m$ and $V_m$ at the beginning and the end of the interval, respectively.)
Let's find the root-mean-square value of this function in the time interval from 0 to \( t' \gg T \), so that the time interval is much longer than one period of the function.

\[
V_{\text{rms}} = \sqrt{\frac{\int_0^{t'} \left( \frac{2V_m}{T} \cdot t - (2N - 1)V_m \right)^2 dt}{t' - 0}} = \\
= \sqrt{\frac{\int_0^T \left( \frac{2V_m}{T} \cdot t - V_m \right)^2 dt}{t'} + \frac{\int_{NT}^{NT+\Delta t'} \left( \frac{2V_m}{T} \cdot t - (2N - 1)V_m \right)^2 dt}{t'}}
\]

where \( N \) is the number of "full" periods in the considered time interval, and \( \Delta t' \) is the time remaining after the last "full" period. The last integral is over a time interval shorter than one period of the function (\( \Delta t' < T \)).

Let's discuss each integral separately beginning with the second one. As time \( t' \) is much longer than one period the second term is approximately zero.

\[
0 \leq \lim_{t' \to \infty} \frac{\int_{NT}^{NT+\Delta t'} \left( \frac{2V_m}{T} \cdot t - (2N - 1)V_m \right)^2 dt}{t'} < \lim_{t' \to \infty} \frac{\int_0^T V_m^2 dt}{t'} = 0
\]
The first term yields

\[
N \int_0^T \left( \frac{2V_m}{T} \cdot t - V_m \right)^2 dt = N \cdot \frac{T}{2V_m} \int x^2 dx
\]

\[
= \lim_{t' \to \infty} \frac{NT}{NT + \Delta t'} \cdot \frac{1}{2V_m} \cdot \frac{x^3}{3} \bigg|_{-V_m}^{V_m} = \frac{V_m^2}{3}
\]

Using our findings in the initial expression for the root-mean-square value of the voltage, we can relate it to the peak value of the voltage.

\[
V_{rms} \approx \sqrt{\frac{V_m^2}{3} + 0} = \frac{V_m}{\sqrt{3}}
\]

Note that similar to the relationship for the sinusoidal voltage, we can assume that the root-mean-square value of the voltage does not depend on time for time intervals much longer than the period of the voltage.