Homework 7

chapter 33: 21, 25, 49, 65
Problem 33.21

An RLC circuit consists of a 150-Ω resistor, a 21-μF capacitor and a 460-mH inductor, connected in series with a 120-V, 60-Hz power supply. (a) What is the phase angle between the current and the applied voltage? (b) Which reaches its maximum earlier, the current or the voltage?

\[ X_L = \omega L = 2\pi \cdot 60\text{Hz} \cdot 0.46\text{H} = 173\Omega \]

\[ X_C = \frac{1}{\omega C} = \frac{1}{2\pi \cdot 60\text{Hz} \cdot 21 \cdot 10^{-6}\text{F}} = 126\Omega \]

For the series connection, the instantaneous voltage across the system is equal to the sum of voltage across each element. Using the phasor diagram, we can find that the phase angle between the voltage (across the system) and the current (through the system) is:

\[ \phi = \arctan \frac{X_L - X_C}{R} = \arctan \frac{174\Omega - 126\Omega}{150\Omega} = 17.4^\circ \]
b) From the definition of phase angle, we find that the initial phase of the voltage is greater than the initial phase of the current.

\[ \delta_V = \delta_I + \phi \]

The voltage leads the current.
Problem 33.25

Draw to scale a phasor diagram showing $Z$, $X_L$, $X_C$, and $\phi$ for an ac series circuit for which $R = 300 \, \Omega$, $C = 11 \, \mu F$, $L = 0.2 \, H$, and $f = \frac{500}{\pi} \, Hz$.

Before drawing the diagram, I will find the complex impedance of each element. The complex impedance of a resistor is equal to its resistance.

$$Z_R = R = 300\Omega$$

It is therefore a real number which I marked in the complex plane on the Re axis.

The complex impedance of an inductor is an imaginary number dependent on the angular frequency of the current and the inductance of the inductor

$$Z_L = i\omega L = 2\pi \cdot \frac{500}{\pi} \cdot 0.2H \cdot i = 200i\Omega$$

This number is on the Im axis. Finally, the complex impedance of the capacitor is also an imaginary number dependent on the angular frequency of the current and the capacitance of the capacitor

$$Z_C = \frac{-i}{\omega C} = \frac{-i}{2\pi \cdot \frac{500}{\pi} s^{-1} \cdot 11 \cdot 10^{-6} F} = -90.9i\Omega$$

This number is also on the Im axis.

Since the elements are connected in series the equivalent complex impedance is equal to the sum of the complex impedance of each element.

$$Z = Z_R + Z_L + Z_C = 300\Omega + 200i\Omega - 90.9i\Omega = (300 + 109.1i)\Omega$$

The complex impedance of each element is indicated in the diagram below. One tic corresponds to $100\Omega$. 

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The diagram also illustrates how to find the equivalent complex impedance. Note that the phase of the complex impedance of the resistor is 0, the phase of the complex impedance of the capacitor is -90° and the phase of the complex impedance of the inductor is 90°.

Supplement. Although it is not asked in the problem, let's see in scale the relationship between the current (common for all elements) and the voltages in the circuit. Let's assume that the current has value of 20 mA and at a certain instant t it is represented by the phasor indicated in the following figure. In the figure one tic corresponds to 10 mA.

The complex voltage across the resistor can be found by multiplying the complex current (through the resistor) by the complex impedance of the resistor

\[ V_R(t) = I(t) \cdot Z_R \]

Consistent with the given values, the absolute value of this voltage is
\[ V_{R,m} = I_m \cdot R = 20mA \cdot 300\Omega = 6V \]

Since the phase of the complex impedance of the resistor is zero, the phase of the voltage across the resistor agrees with the phase of the current. In the figure, one tic corresponds to 1V.

The complex voltage across the inductor can be found by multiplying the complex current (through the inductor) by the complex impedance of the inductor

\[ V_L(t) = I(t) \cdot Z_L \]

Consistent with the given values, the absolute value of this voltage is

\[ V_{L,m} = I_m \cdot X_L = 20mA \cdot 200\Omega = 4V \]

Since the phase of the complex impedance of the inductor is 90°, the voltage across the inductor leads with the current by a 90° phase angle.

Finally, the complex voltage across the capacitor can be found by multiplying the complex current (through the capacitor) by the complex impedance of the capacitor

\[ V_C(t) = I(t) \cdot Z_C \]

Consistent with the given values, the absolute value of this voltage is

\[ V_{C,m} = I_m \cdot X_C = 20mA \cdot 91\Omega = 1.8V \]

Since the phase of the complex impedance of the capacitor is -90°, the voltage across the capacitor lags behind the current by 90°.

If we want to find the complex voltage across the entire system, we can add the complex voltages across all three elements (they are connected in series). I indicated this operation on the phasor diagram. That voltage can also be
found by multiplying the current by the equivalent impedance of the system. Notice that the phase difference between the voltage across the system and the current through the system is equal to the phase angle of the system (the phase of the equivalent impedance). The absolute value of the voltage is

\[
V_m = I_m \cdot Z = 20\text{mA} \cdot \sqrt{(300\Omega)^2 + (109\Omega)^2} \approx 6.4\text{V}
\]
Problem 33.49

The resistor in the Figure P33.49 represents the midrange speaker in a three-speaker system. Assume its resistance to be constant at 8 Ω. The source represent an audio amplifier producing signals of uniform amplitude ΔV_{in} = 10V al all frequencies. The inductor and capacitor are to function as a bandpass filter with \( \frac{\Delta V_{out}}{\Delta V_{in}} = \frac{1}{2} \) at 200 Hz and 4,000 Hz. (a) Determine the required values of L and C. (b) Find the maximum value of the ratio \( \frac{\Delta V_{out}}{\Delta V_{in}} \). (c) Find the frequency \( f_0 \) at which the ratio has its maximum value. (d) Find the phase shift between \( \Delta V_{in} \) and \( \Delta V_{out} \) at 200 Hz, at \( f_0 \), and at 4000 Hz. (e) Find the average power transferred to the speaker at 200 Hz, at \( f_0 \), and at 4000 Hz. (f) Treating the filter as a resonant circuit, find its quality factor.

Complex current in the series RLC circuit is determined by the input complex voltage and the complex impedance of the circuit

\[
\mathbf{\mathbf{j} = \frac{\mathbf{V}_{in}}{\mathbf{Z}}} = \frac{\mathbf{V}_{in}}{R + i\omega L + \frac{1}{i\omega C}} = \frac{\mathbf{V}_{in}}{R + i\left(\omega L - \frac{1}{\omega C}\right)}
\]

Complex voltage across the speaker is related to the complex current through the speaker.
\[ \mathcal{V}_{\text{out}} = G \cdot R = \frac{R}{R + i \cdot \left( \omega L - \frac{1}{\omega C} \right)} \cdot \mathcal{V}_{\text{in}} = \frac{R^2 - i \cdot \left( \omega L - \frac{1}{\omega C} \right) R}{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2} \cdot \mathcal{V}_{\text{in}} \]

The number relating complex output voltage to complex input voltage is called the complex gain of the circuit.

\( \Delta V_{\text{out}} = |\mathcal{V}_{\text{out}}| = \frac{R}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}} \cdot \Delta V_{\text{in}} \)

The number relating the amplitudes of voltage is called the gain of the circuit. Gain \( G \) is a function of frequency

\[ G = \frac{R}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}} \]

It assumes a particular value at specific frequencies, determined by the difference between capacitive reactance and inductive reactance

\[ \omega_1 L - \frac{1}{\omega_1 C} = -\sqrt{\frac{1}{G^2} - 1} \cdot R \quad \text{and} \quad \omega_2 L - \frac{1}{\omega_2 C} = \sqrt{\frac{1}{G^2} - 1} \cdot R \]
Solving simultaneously for the unknowns

\[ L = \frac{R \sqrt{\frac{1}{G^2} - 1}}{\omega_2 - \omega_1} = \frac{R \sqrt{\frac{1}{G^2} - 1}}{2\pi(f_2 - f_1)} = \frac{8\Omega \cdot \sqrt{4 - 1}}{2\pi \cdot (4000\text{Hz} - 200\text{Hz})} = 580\mu\text{H} \]

\[ C = -\frac{\omega_1 - \omega_2}{R \cdot \omega_1 \omega_2 \sqrt{\frac{1}{G^2} - 1}} = \frac{f_2 - f_1}{2\pi R \cdot f_1 f_2 \sqrt{\frac{1}{G^2} - 1}} = \frac{4000\text{Hz} - 200\text{Hz}}{2\pi \cdot 8\Omega \cdot 200\text{Hz} \cdot 4000\text{Hz} \cdot \sqrt{4 - 1}} = 54.6\mu\text{F} \]

(b) The maximum gain occurs when the sum of inductive reactance and capacitive reactance is zero

\[ \omega L - \frac{1}{\omega C} = 0 \]

from which

\[ G = \frac{R}{\sqrt{R^2 + 0^2}} = 1 \]

(c) The sum of inductive reactance and capacitive reactance is zero at the resonant frequency of this circuit

\[ f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi \cdot \sqrt{(5.8 \cdot 10^{-4}\text{H}) \cdot (5.46 \cdot 10^{-5}\text{F})}} = 894\text{Hz} \]
(d) Complex gain also includes the phase relation. The phase difference $\alpha$, between the output voltage and the input voltages is

$$\alpha = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) = -\arctan\frac{2\pi f L - \frac{1}{2\pi f C}}{R}$$

Therefore at 200 Hz its value is

$$\alpha_{200\text{Hz}} = -\arctan\frac{1}{2\pi \cdot 200\text{Hz} \cdot (5.8 \cdot 10^{-4} \text{H})} = 60^\circ$$

(The output voltage leads the input voltage.)

At 894 Hz its value is

$$\alpha_{894\text{Hz}} = -\arctan\frac{0}{8\Omega} = 0^\circ$$

(The output voltage is in phase with the input voltage.)

At 4000 Hz its value is

$$\alpha_{4000\text{Hz}} = -\arctan\frac{1}{2\pi \cdot 4000\text{Hz} \cdot (5.46 \cdot 10^{-5} \text{H})} = -60^\circ$$

(The output voltage lags behind the input voltage.)
(e) The average power delivered to an element depends on the rms values of the voltage across the element and the current through the element. Since the speaker acts like a resistor, Ohm’s law can be used to relate the rms current (through the speaker) to the voltage applied to the speaker. Knowing the gain of the circuit the power can be directly related to the amplitude of the input voltage

\[ P = I_{\text{rms}} V_{\text{rms}} \cos \varphi = \frac{\Delta V_{\text{out}}}{R\sqrt{2}} \cdot \frac{\Delta V_{\text{out}}}{\sqrt{2}} \cdot \cos \varphi = \frac{(G \cdot \Delta V_{\text{in}})^2}{2R} \]

Hence

\[
P_{200} = \frac{(0.5 \cdot 10 V)^2}{2 \cdot 8 \Omega} = 1.56 \text{W}
\]

\[
P_{894} = \frac{(1 \cdot 10 V)^2}{2 \cdot 8 \Omega} = 6.25 \text{W}
\]

\[
P_{4000} = \frac{(0.5 \cdot 10 V)^2}{2 \cdot 8 \Omega} = 1.56 \text{W}
\]

(f) Parameters of the RLC circuit (resistance, capacitance and inductance) affect the quality factor

\[
Q = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R} = \frac{2\pi \cdot 894 \text{Hz} \cdot (5.8 \cdot 10^{-4} \text{H})}{8 \Omega} = 0.41
\]
**Problem 33.65**

Figure 33.69a shows a parallel RLC circuit, and the corresponding phasor diagram is given in Figure 33.29b.

The instantaneous voltage (and rms voltage) across each of the three circuit elements is the same, and each is in phase with the current through the resistor. The currents in the capacitor and the inductor lead or lag behind the current in the resistor, as shown in Figure 33.66b. (a) Show that the rms current delivered by the source is given by

$$I_{rms} = V_{rms} \sqrt{\frac{1}{R^2} + \left(\frac{\omega C - \frac{1}{\omega L}}{2}\right)^2}$$

(b) Show that the phase angle between the voltage and the current is given by

$$\tan \varphi = -R \left(\frac{1}{X_C} - \frac{1}{X_L}\right)$$

**In fact, we are asked to find the impedance and the phase angle for this system of elements connected in parallel. It will be easier to analyze the complex impedance. For elements connected in parallel the equivalent complex impedance of the system is equal to the sum of inverse impedance of each element**

$$\frac{1}{Z_{eq}} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C = \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)$$

or
\[ \begin{align*}
Z_{eq} &= \frac{1}{\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)} = \frac{1}{R} - i\left(\omega C - \frac{1}{\omega L}\right) \\
&= \frac{1}{\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)}^2
\end{align*} \]

(a) Therefore

\[ \begin{align*}
I_{rms} &= \frac{1}{\sqrt{2}} \cdot I_m = \frac{1}{\sqrt{2}} \cdot |I| = \frac{1}{\sqrt{2}} \cdot \left|\frac{\psi}{Z_{eq}}\right| = \frac{V_m}{\sqrt{2} \cdot \left|Z_{eq}\right|} = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2} \cdot V_{rms}
\end{align*} \]

(b) and

\[ \begin{align*}
tan \varphi &= \frac{\text{Im} Z_{eq}}{\text{Re} Z_{eq}} = \frac{-\left(\omega C - \frac{1}{\omega L}\right)}{\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)} = \frac{-\left(\omega C - \frac{1}{\omega L}\right)}{\frac{1}{R}} = -R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)
\end{align*} \]